MAT 313 Abstract Algebra 2002
Midterm Solutions

Name:____

I.D.:____

1	2	3	4	EC	TOTAL
10 points	10 points	20 points	10 points	10 points	50 points

Problem 1 (i) (4 points) Find the order of the group $U(20)$.

(ii) (6 points) Find the order of each element in the group $U(20)$ and determine whether $U(20)$ is cyclic or not.

Solution

(i) $U(20) = \{1, 3, 7, 9, 11, 13, 17, 19\}$, so that $|U(20)| = 8$.

(ii) $|3| = |7| = |13| = |17| = 4$ and $|9| = |11| = |19| = 2$, so that $U(20)$ is not cyclic since it does not contain element of order 8.

Problem 2 (i) (4 points) Let $G = \langle a \rangle$ be a cyclic group of order n, and let k be a positive integer. Find the order of the element a^k.

(ii) (6 points) Let $G = \mathbb{Z}_{90000}$. Using part (i), find all elements of order 9 in G.

Solution

(i) $|a^k| = n/\gcd(k, n)$

(ii) There are $6 = \phi(9)$ elements of order 9, they are elements $k \in \mathbb{Z}_{90000}$ such that $\gcd(k, 90000) = 90000/9 = 10000$. Explicitly, $k = 10000, 20000, 40000, 50000, 70000, 80000$.

Problem 3 Let G be a group and let $x, y \in G$ be elements such that $|y| = 2$, $x \neq e$ and

$$yxy = x^2.$$

(i) (3 points) Using the above equation show that $(yx)^2 = x^3$.

(ii) (3 points) Using the above equation show that $yx = xyx^{-1}$ (*Hint*: mupliply the above equation by x^{-1} from the right and use $|y| = 2$.)

(iii) (8 points) Prove that $|xyx^{-1}| = |y| = 2$ and find $|yx|$ using part (ii).

iv (6 points) Find $|x|$ using parts (i) and (iii).

Solution

(i) Multiply $yxy = x^2$ by x *from the right* to get $x^3 = yxyx = (yx)^2$.

(ii) Following hint, we get $yxyx^{-1} = x$. Multiplying this equation by y *from the left* and using $y^2 = e$ we get $xyx^{-1} = yx$.

(iii). We have $(xyx^{-1})^2 = xyx^{-1}xyx^{-1} = xy^2x^{-1} = xx^{-1} = e$. Next, $xyx^{-1} \neq e$. Indeed, $xyx^{-1} = e$ implies that $xy = x$ or, by cancelling, $y = e$, which contradicts $|y| = 2$. Thus $|xyx^{-1}| = 2$ and, by (ii), $|yx| = 2$.

(iv) Since $|yx| = 2$ we get, using (i), that $x^3 = e$. Since $x \neq e$ we get $|x| = 3$.

Problem 4 Consider the permutation

$$\sigma = (1235)(2467)$$

(i) (2 points) Find $\sigma(4)$ and $\sigma^2(4)$.
(ii) (2 points) Find σ^{-1}.
(iii) (3 points) Write σ as a product of disjoint cycles (double-check your answer with part (i)).
(iv) (3 points) Find the order of σ.

Solution

(i) $\sigma(4) = 6,\ \sigma^2(4) = \sigma(6) = 7$
(ii) $\sigma^{-1} = (7642)(5321)$
(iii) $\sigma = (1246735)$
(iv) $|\sigma| = 7$.

Extra Credit (i) (5 points) Let G be a group, let H be its subgroup, and let $a \in G$ be an element of order n. Prove that if $a^m \in H$ and m and n are relatively prime, then $a \in H$.
(ii) (5 points) Let $x, y \in G$ and $xy \in C(x)$, the centralizer of x in G. Prove that $xy = yx$.

Solution

(i) Since $\gcd(m, n) = 1$, there exist integers s and t such that $sm + tn = 1$. Now

$$a = a^{sm+tn} = a^{sm}a^{tn} = (a^n)^t(a^m)^s = (a^m)^s \in H$$

using $a^n = e$, $a^m \in H$ and the fact that H is a subgroup.

(ii) Since $C(x) = \{g \in G \,|\, gx = xg\}$, condition $xy \in C(x)$ means that $xyx = xxy$. Cancelling by x from the left gives $yx = xy$.