

## MAT 126 Calculus B Spring 2006 Practice Midterm II Solutions

Answer each question in the space provided and on the reverse side of the sheets. Show your work whenever possible. Unless otherwise indicated, **answers without justification will get little or no partial credit!** Cross out anything that grader should ignore and circle or box the final answer. The actual exam will contain 5 problems. This practice test contains more problems to give you more practice.

1. Evaluate the following definite integrals

(a)

$$\int_0^{13} \frac{2}{(2x+1)^{\frac{2}{3}}} dx$$

*Solution.* Substitution  $u = 2x + 1$  gives  $du = 2dx$  and  $x = 0$  corresponds to  $u = 1$  and  $x = 13$  — to  $u = 27$ . Thus by the substitution rule,

$$\begin{aligned} \int_0^{13} \frac{2}{(2x+1)^{\frac{2}{3}}} dx &= \int_1^{27} \frac{1}{u^{\frac{2}{3}}} du = 3u^{\frac{1}{3}} \Big|_1^{27} \\ &= 3(27)^{\frac{1}{3}} - 3 = 3 \cdot 3 - 3 = 6. \end{aligned}$$

(b)

$$\int_0^{\frac{\pi}{2}} e^{\sin x} \cos x dx$$

*Solution.* Substitution  $u = \sin x$  gives  $du = \cos x dx$  and  $x = 0$  corresponds to  $u = 0$  and  $x = \pi/2$  — to  $u = 1$ . Thus by the substitution rule,

$$\int_0^{\frac{\pi}{2}} e^{\sin x} \cos x dx = \int_0^1 e^u du = e^u \Big|_0^1 = e - 1.$$

(c)

$$\int_0^1 x^4(1+x^5)^{20} dx$$

*Solution.* Substitution  $u = 1 + x^5$  gives  $du = 5x^4 dx$  and  $x = 0$  corresponds to  $u = 1$  and  $x = 1$  — to  $u = 2$ . Thus by the substitution rule,

$$\int_0^1 x^4(1+x^5)^{20} dx = \frac{1}{5} \int_1^2 u^{20} du = \frac{u^{21}}{5 \cdot 21} \Big|_1^2 = \frac{2^{21} - 1}{105}.$$

(d)

$$\int_0^1 \tan^{-1} x \, dx$$

*Solution.* We use integration by parts with  $u = \tan^{-1} x$  and  $dv = dx$ . We have  $du = \frac{dx}{1+x^2}$  and  $v = x$ , so that using  $\tan^{-1}(1) = \frac{\pi}{4}$ , we get

$$\begin{aligned} \int_0^1 \tan^{-1} x \, dx &= \int_0^1 u \, dv = uv \Big|_0^1 - \int_0^1 v \, du \\ &= \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2} \, dx. \end{aligned}$$

To evaluate the remaining integral, we use the substitution  $u = 1 + x^2$ , so that  $du = 2x \, dx$  and  $x = 0$  corresponds to  $u = 1$  and  $x = 1$  — to  $u = 2$ . Thus

$$\int_0^1 \frac{x}{1+x^2} \, dx = \frac{1}{2} \int_1^2 \frac{du}{u} = \frac{\ln u}{2} \Big|_1^2 = \frac{\ln 2}{2}.$$

Therefore, we get

$$\int_0^1 \tan^{-1} x \, dx = \frac{\pi}{4} - \frac{\ln 2}{2}.$$

(e)

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\sin t| \, dt.$$

*Solution.* This is the integral over the symmetric interval  $[-\frac{\pi}{2}, \frac{\pi}{2}]$  of the even function

$$f(t) = |\sin t|.$$

Since  $\sin t$  is non-negative on  $[0, \frac{\pi}{2}]$ , we get

$$\begin{aligned} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\sin t| \, dt &= 2 \int_0^{\frac{\pi}{2}} |\sin t| \, dt = 2 \int_0^{\frac{\pi}{2}} \sin t \, dt \\ &= -2 \cos t \Big|_0^{\frac{\pi}{2}} = -2(\cos \frac{\pi}{2} - \cos 0) = 2. \end{aligned}$$

(f)

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1+t^2)^2 \sin^5 t \, dt.$$

*Solution.* It is an integral over the symmetric interval  $[-\frac{\pi}{2}, \frac{\pi}{2}]$  of the odd function

$$f(t) = (1 + t^2)^2 \sin^5 t.$$

(Verify that  $f(-t) = -f(t)$ !). Therefore,

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + t^2)^2 \sin^5 t \, dt = 0.$$

**2.** Evaluate the following indefinite integrals

(a)

$$\int x^3 e^{x^4} \, dx$$

*Solution.* Setting  $u = x^4$ , we get  $du = 4x^3 dx$ , so by the substitution rule,

$$\int x^3 e^{x^4} \, dx = \frac{1}{4} \int e^u \, du = \frac{1}{4} e^u + C = \frac{1}{4} e^{x^4} + C.$$

(b)

$$\int t e^t \, dt$$

*Solution.* We use integration by parts with  $u = t$  and  $dv = e^t dt$ . We have  $du = dt$  and  $v = e^t$ , so that

$$\int t e^t \, dt = \int u \, dv = uv - \int v \, du = t e^t - \int e^t \, dt = t e^t - e^t + C.$$

(c)

$$\int x^2 \cos x \, dx$$

*Solution.* We use integration by parts with  $u = x^2$  and  $dv = \cos x \, dx$ . We have  $du = 2x \, dx$  and  $v = \sin x$ , so that

$$\int x^2 \cos x \, dx = \int u \, dv = uv - \int v \, du = x^2 \sin x - 2 \int x \sin x \, dx.$$

For the remaining integral we again use integration by parts with  $u = 2x$  and  $dv = \sin x \, dx$ , so that  $du = 2 \, dx$  and  $v = -\cos x$ . We have

$$\begin{aligned} 2 \int x \sin x \, dx &= \int u \, dv = uv - \int v \, du = -2x \cos x + 2 \int \cos x \, dx \\ &= -2x \cos x + 2 \sin x + C, \end{aligned}$$

so that

$$\int x^2 \cos x \, dx = x^2 \sin x + 2x \cos x - 2 \sin x + C.$$

(Double-check the answer by differentiating!)

(d)

$$\int \cos(\sqrt{x}) dx$$

*Solution.* First, we use the substitution rule with  $t = \sqrt{x}$ , so that  $dt = \frac{1}{2\sqrt{x}} dx$ , or  $dx = 2\sqrt{x} dt = 2t dt$ . We get

$$\int \cos(\sqrt{x}) dx = 2 \int t \cos t dt.$$

To evaluate this integral, we use integration by parts with  $u = 2t$  and  $dv = \cos t dt$ . We have  $du = 2 dt$  and  $v = \sin t$ , so that

$$\begin{aligned} 2 \int t \cos t dt &= \int u dv = uv - \int v du \\ &= 2t \sin t - 2 \int \sin t dt = 2t \sin t + 2 \cos t + C. \end{aligned}$$

Finally, remembering that  $t = \sqrt{x}$ , we get

$$\int \cos(\sqrt{x}) dx = 2\sqrt{x} \sin \sqrt{x} + 2 \cos \sqrt{x} + C.$$

3. Evaluate the following indefinite integrals

(a)

$$\int \frac{1}{x^2} \ln x dx$$

*Solution.* Here we use integration by parts with  $u = \ln x$  and  $dv = \frac{1}{x^2} dx$ , so that

$$du = \frac{1}{x} dx \quad \text{and} \quad v = -\frac{1}{x}.$$

(Note that substitution rule with  $u = \ln x$  does not simplify the integral since in the denominator we have  $x^2$ ; if it was  $x$ , then the substitution rule would work.) Thus we have

$$\begin{aligned} \int \frac{1}{x^2} \ln x dx &= \int u dv = uv - \int v du \\ &= -\frac{\ln x}{x} + \int \frac{1}{x^2} dx = -\frac{\ln x}{x} - \frac{1}{x} + C. \end{aligned}$$

(b)

$$\int \frac{1}{x} (\ln x)^2 dx$$

*Solution.* Here we use the substitution rule with  $u = \ln x$  and  $du = \frac{1}{x}dx$  (since we have  $x$  in the denominator).

Therefore,

$$\int \frac{1}{x} (\ln x)^2 dx = \int u^2 du = \frac{1}{3}u^3 + C = \frac{1}{3}(\ln x)^3 + C.$$

(c)

$$\int x^7 \ln x dx$$

*Solution.* Here we use integration by parts with  $u = \ln x$  and  $dv = x^7 dx$ , so that

$$du = \frac{1}{x}dx \quad \text{and} \quad v = \frac{1}{8}x^8.$$

Thus we have

$$\begin{aligned} \int x^7 \ln x dx &= \int u dv = uv - \int v du \\ &= \frac{1}{8}x^8 \ln x - \frac{1}{8} \int x^8 \frac{dx}{x} = \frac{1}{8}x^8 \ln x - \frac{1}{64}x^8 + C. \end{aligned}$$

4. Evaluate the following indefinite integrals

(a)

$$\int \frac{2x^2}{x^2 + 1} dx$$

*Solution.* We have

$$\frac{2x^2}{x^2 + 1} = 2 - \frac{2}{x^2 + 1}$$

(either by doing the long division, or by writing  $2x^2 = 2x^2 + 2 - 2 = 2(x^2 + 1) - 2$ , and dividing both terms by  $x^2 + 1$ ). Thus

$$\int \frac{2x^2}{x^2 + 1} dx = \int \left( 2 - \frac{2}{x^2 + 1} \right) dx = 2x - 2 \tan^{-1} x + C.$$

(b)

$$\int \frac{2x}{x^2 + 1} dx$$

*Solution.* To evaluate this integral, we use the substitution  $u = 1 + x^2$ , so that  $du = 2x dx$  (compare with the last integral in problem 2 (d)). We have

$$\int \frac{2x}{x^2 + 1} dx = \int \frac{du}{u} = \ln |u| + C = \ln(1 + x^2) + C.$$

5. (a) Write a formula for  $\tan x$  in terms of  $\sin x$  and  $\cos x$ .

*Solution.*

$$\tan x = \frac{\sin x}{\cos x}.$$

- (b) Evaluate

$$\int \tan x \, dx$$

*Solution.* Using part (a) we have

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx,$$

which suggests the substitution  $u = \cos x$ . We have  $du = -\sin x \, dx$ , so that

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = - \int \frac{du}{u} = -\ln |u| + C = -\ln |\cos x| + C.$$

6. Evaluate

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{x}{\sin^2 x} \, dx$$

*Solution.* We use integration by parts with  $u = x$  and

$$dv = \frac{1}{\sin^2 x} \, dx,$$

so that  $du = dx$  and  $v = -\cot x$ . We have, using that anti-derivative of  $\cot x$  is  $\ln |\sin x|$  (compare with part (a)),

$$\begin{aligned} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{x}{\sin^2 x} \, dx &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} u \, dv = uv \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} v \, du \\ &= -x \cot x \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot x \, dx \\ &= \frac{\pi}{4} + \ln |\sin x| \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \frac{\pi}{4} + \frac{1}{2} \ln 2, \end{aligned}$$

where we have used that  $\cot \frac{\pi}{4} = 1$ ,  $\cot \frac{\pi}{2} = 0$  and  $\sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$ ,  $\sin \frac{\pi}{2} = 1$ .

7. (a) Set

$$f(x) = \int_1^{x^2} \sin t^3 \, dt + x^3$$

Find  $f(1)$  and  $f'(x)$ .

*Solution.* First,  $f(1) = 1^3 = 1$ , since in this case the interval of integration shrinks to a point and the integral

is zero. Second, we get by the FTC and the chain rule, setting  $u = x^2$ ,

$$\begin{aligned} f'(x) &= \frac{du}{dx} \frac{d}{du} \left( \int_1^u \sin t^3 dt \right) \Big|_{u=x^2} + 3x^2 \\ &= 2x (\sin u^3) \Big|_{u=x^2} + 3x^2 = 2x \sin x^6 + 3x^2. \end{aligned}$$

(b) Set

$$f(x) = \int_{\sqrt{x}}^{x-2} \tan^2 t dt$$

Find  $f(4)$  and  $f'(x)$

*Solution.* We have

$$f(4) = \int_{\sqrt{4}}^{4-2} \tan^2 t dt = \int_2^2 \tan^2 t dt = 0.$$

To find  $f'(x)$ , we write

$$\begin{aligned} f(x) &= \int_{\sqrt{x}}^{x-2} \tan^2 t dt = \int_{\sqrt{x}}^0 \tan^2 t dt + \int_0^{x-2} \tan^2 t dt \\ &= - \int_0^{\sqrt{x}} \tan^2 t dt + \int_0^{x-2} \tan^2 t dt, \end{aligned}$$

and apply the FTC and the chain rule (for the first integral we use  $u = \sqrt{x}$ , and for the second integral we use  $u = x - 2$ ).

We get

$$\begin{aligned} f'(x) &= - \frac{du}{dx} \frac{d}{du} \left( \int_0^u \tan^2 t dt \right) \Big|_{u=\sqrt{x}} + \frac{du}{dx} \frac{d}{du} \left( \int_0^u \tan^2 t dt \right) \Big|_{u=x-2} \\ &= - \frac{1}{2\sqrt{x}} \tan^2(\sqrt{x}) + \tan^2(x-2). \end{aligned}$$