Section 8.4

2.

$a)\ \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 8 \qquad \sum a_n$ is divergent.

$b)\ \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0.8 \qquad \sum a_n$ is convergent.

$c)\ \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1 \qquad \sum a_n$ may or may not convergent, that is,

we cannot determine that the series is convergent or divergent using this test.

4.

$$-\frac{1}{3}+\frac{2}{4}-\frac{3}{5}+\frac{4}{6}-\frac{5}{7}+\cdots$$

Let the nth term of the series is a_n. Then

$$a_n = (-1)^n \frac{n}{n+2} \quad \textit{Thus} \quad \lim_{n\to\infty} a_n \quad \textit{does not exist.}$$

Therefore, $\sum a_n$ does not converge.

12.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} \qquad (|error| < 0.001)$$

$$\textit{Let } b_n = \frac{1}{n^4},\ \textit{then } b_n < b_{n+1} \ \textit{and} \ \lim_{n\to\infty} b_n = 0.$$

Hence, the series is convergent.

$$\textit{Moreover}, \quad |R_n| \le b_{n+1}$$

$$\textit{Then if } \frac{1}{n^4} < 0.001,\ \textit{then } n > \frac{1}{\sqrt[4]{0.001}} \approx 5.623413$$

Therefore, we need 5 terms.

22.

$$\sum_{n=0}^{\infty}\frac{(-3)^n}{n!} \quad Let \quad a_n = \frac{(-3)^n}{n!}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{3^{n+1}}{(n+1)!}\cdot\frac{n!}{3^n} = \lim_{n\to\infty}\frac{3}{n+1} = 0$$

Therefore, $\sum a_n$ is absolutely convergent.

24.

$$\sum_{n=1}^{\infty}(-1)^n\frac{n}{n^2+1}, \quad Let \quad a_n = (-1)^n\frac{n}{n^2+1}$$

$$|a_n| = \frac{n}{n^2+1} \geq \frac{n}{2n^2} = \frac{1}{2n}, \quad for\ all\ n \geq 1$$

$\sum_{n=1}^{\infty}\frac{1}{2n}$ is divergent, *so* $\sum a_n$ is also divergent by the Comparision Test.

Therefore, the series does not converge absolutely. (However, the series is convergent by the test for alternating series.)

30.

$$a_1 = 1, \ a_{n+1} = \frac{2+\cos n}{\sqrt{n}}\cdot a_n$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{2+\cos n}{\sqrt{n}} \leq \lim_{n\to\infty}\frac{3}{\sqrt{n}} = 0, \quad So\ that\ \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0$$

Therefore, the series is absolutely convergent.

34.

$$\sum a_n, \quad r_n = \frac{a_{n+1}}{a_n}, \ and\ \lim_{n\to\infty} r_n = L < 1$$

$$R_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$$

a) Prove that $if \quad r_n > r_{n+1}, \ and\ r_{n+1} < 1, \ then\ R_n \leq \frac{a_{n+1}}{1-r_{n+1}}$

$$
\begin{aligned}
R_n &= a_{n+1} + a_{n+2} + a_{n+3} + a_{n+4} + \cdots \\
&= a_{n+1} + \frac{a_{n+2}}{a_{n+1}} \cdot a_{n+1} + \frac{a_{n+3}}{a_{n+2}} \cdot \frac{a_{n+2}}{a_{n+1}} \cdot a_{n+1} + \frac{a_{n+4}}{a_{n+3}} \cdot \frac{a_{n+3}}{a_{n+2}} \cdot \frac{a_{n+2}}{a_{n+1}} \cdot a_{n+1} + \cdots \\
&= a_{n+1}(1 + \frac{a_{n+2}}{a_{n+1}} + \frac{a_{n+3}}{a_{n+2}} \cdot \frac{a_{n+2}}{a_{n+1}} + \frac{a_{n+4}}{a_{n+3}} \cdot \frac{a_{n+3}}{a_{n+2}} \cdot \frac{a_{n+2}}{a_{n+1}} + \cdots) \\
&= a_{n+1}(1 + r_{n+1} + r_{n+2} \cdot r_{n+1} + r_{n+3} \cdot r_{n+2} \cdot r_{n+1} + \cdots) \\
&\le a_{n+1}(1 + r_{n+1} + r_{n+1}^2 + r_{n+1}^3 + \cdots) ------------ (*) \; . \\
&= a_{n+1} \cdot \frac{1}{1 - r_{n+1}}
\end{aligned}
$$

Remark: $\left| r_{n+1} \right| < 1$, and $\{r_{n+1}\}$ is decireasing, so (*) is convergent.

Therefore, $R_n \le \dfrac{a_{n+1}}{1 - r_{n+1}}$

b) Prove that *if* $\{r_n\}$ *is the increasing sequence, then* $R_n \le \dfrac{a_{n+1}}{1 - L}$

$$
\begin{aligned}
R_n &= a_{n+1} + a_{n+2} + a_{n+3} + a_{n+4} + \cdots \\
&= a_{n+1} + \frac{a_{n+2}}{a_{n+1}} \cdot a_{n+1} + \frac{a_{n+3}}{a_{n+2}} \cdot \frac{a_{n+2}}{a_{n+1}} \cdot a_{n+1} + \frac{a_{n+4}}{a_{n+3}} \cdot \frac{a_{n+3}}{a_{n+2}} \cdot \frac{a_{n+2}}{a_{n+1}} \cdot a_{n+1} + \cdots \\
&= a_{n+1}(1 + \frac{a_{n+2}}{a_{n+1}} + \frac{a_{n+3}}{a_{n+2}} \cdot \frac{a_{n+2}}{a_{n+1}} + \frac{a_{n+4}}{a_{n+3}} \cdot \frac{a_{n+3}}{a_{n+2}} \cdot \frac{a_{n+2}}{a_{n+1}} + \cdots) \\
&= a_{n+1}(1 + r_{n+1} + r_{n+2} \cdot r_{n+1} + r_{n+3} \cdot r_{n+2} \cdot r_{n+1} + \cdots) \\
&\le a_{n+1}(1 + L + L^2 + L^3 + \cdots) \\
&= a_{n+1} \cdot \frac{1}{1 - L}
\end{aligned}
$$

Remark: $\{r_n\}$ is increasing sequence, so that $r_n \le L$ *for all n.*

Therefore, $R_n \le \dfrac{a_{n+1}}{1 - L}$