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MAE 301/501
Class Notes – 10/30/08

Some homework problems were initially discussed in class:
Homework #5 Review

(3) Give an example of a rational function that has a **horizontal asymptote** at $y = b$, for some $b \epsilon \mathbf{R}$, and such that, for some *a* in the domain of *f*, *f(a)* = b. Include a clear graph of the function, appropriately scaled, on labeled coordinate axes.

Definition

- We say that *b* is a **horizontal asymptote** of *f(x)* if $\lim_{x\to\infty} f(x) = b$. (It is appropriate to follow the definition of an asymptote with the definition of a limit.)
- The **limit** of *f(x)* as *x* approaches a is *L*
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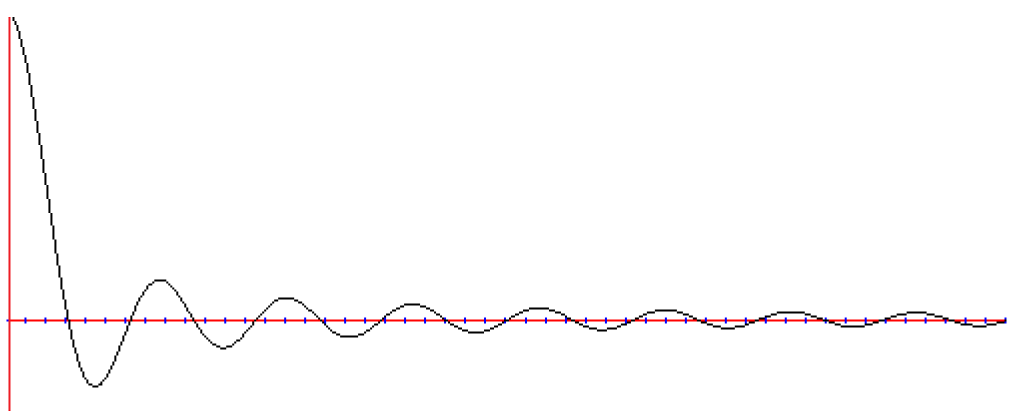
$$\lim_{x\to a} f(x) = L$$

$\leftrightarrow$ given $\varepsilon > 0$, $\exists\delta > 0$ such that $0 < |x - a| < \delta$ implies that $|f(x) - L| < \varepsilon$.

In this problem, we were asked to find a rational function that intersects its horizontal asymptote.

<u>Example</u> Sergeant Tom's example of an oscillating function is *f(x)* = $\frac{\sin(x)}{x}$ when $x \neq 0$



We can see that this function satisfies the properties given in this homework problem.

1) This function has a horizontal asymptote at *y = 0* because $\lim_{x\to\infty} \frac{\sin(x)}{x} = 0$.
2) Let $x = 2\pi$, then $f(2\pi) = \frac{\sin(2\pi)}{2\pi} = \frac{0}{2\pi} = 0$.

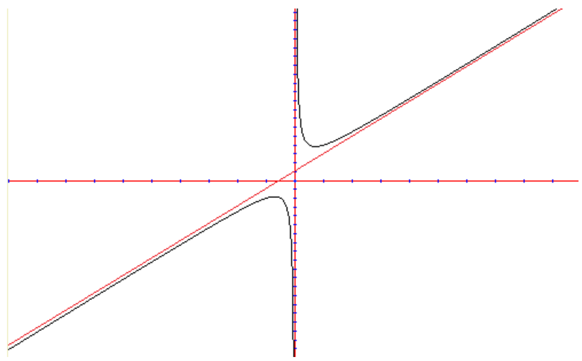
(4) In class we looked at the rational function $f(x) = \frac{2x^2+x+1}{x}$, which has a **slant asymptote**.

Definition

- The function $f(x)$ has a **slant asymptote**, mx $+ b$, as $x \to \infty$,

$$|f(x) - (mx + b)| \to 0.$$

In other words, $mx + b$ is a **slant asymptote** if whenever x gets arbitrarily large, $f(x)$ gets arbitrarily close to the line $mx + b$.



In this figure, the slant asymptote is represented by a red line.

A method discussed in class to find the slant asymptote is taking the limit

$$\lim_{x\to\infty} \frac{2x^2 + x + 1}{x}$$

$$= \lim_{x\to\infty} \frac{2x^2}{x} + \frac{x}{x} + \frac{1}{x}$$

$$= \lim_{x\to\infty} 2x + 1 + \frac{1}{x} = \lim_{x\to\infty} 2x + 1 \qquad since\, \frac{1}{x} \to 0$$

Let $g(x) = 2x + 1$. We can conclude that $\lim_{x\to\infty} \frac{2x^2+x+1}{x} - (2x + 1) = 0$. Hence $g(x) = 2x + 1$ is the slant asymptote of $f(x)$.

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(5) In class some students wrote:

$$\frac{(x+4)(x-7)^2(x+10)}{(x+1)(x-7)(x+10)^2} = \frac{(x+4)(x-7)}{(x+1)(x+10)}$$

(a) Explain the extent to which this equality is correct, and explain how it is incorrect.

i) This equality is correct if we express this algebraically.

$$\frac{(x+4)(x-7)^2(x+10)}{(x+1)(x-7)(x+10)^2} = \frac{(x+4)(x-7)\cancel{(x-7)(x+10)}}{(x+1)\cancel{(x-7)(x+10)}(x+10)} = \frac{(x+4)(x-7)}{(x+1)(x+10)}$$

Therefore, both sides are algebraically equivalent.

ii) On the other hand, this equality is not correct when it represents a function.

On the left hand side, this function does not exist when $x \neq 7$ because

$$\frac{(7+4)(7-7)^2(7+10)}{(7+1)(7-7)(7+10)^2} = \frac{11 * 0^2 * 17}{8 * 0 * 17^2} = \frac{11 * 0^2 * 17}{0}.$$

On the right hand side, when $x \neq 7$

$$\frac{(7+4)(7-7)}{(7+1)(7+10)} = \frac{11 * 0}{8 * 70} = 0$$

Therefore, the left hand side is not equivalent to the right hand side.

The professor handed out the Regents Exams and we discussed **Question #19** in **Part II**

The price of seven race cars sold last week are listed in the table below.

Price per Race Car	**Number of Race Cars**
$126,000	1
$140,000	2
$180,000	1
$400,000	2
$819,000	1

19c) "State which of these measures of central tendency best represents the value of the seven race cards?"

The mean and median can be calculated below:

$$mean = \frac{(126{,}000 * 1)(140{,}000 * 2)(180{,}000 * 1)(400{,}000 * 2)(819{,}000 * 1)}{7} = \$315{,}000$$

~~\$126,000 \$140,000 \$140,000~~ \$180,000 ~~\$400,000 \$400,000 \$819,000~~

Since the median is the middle number of the set listed in order…

$$median = \$180{,}000$$

One student said the measure that best represents the value of the seven cars is the median. The student said that since the median, \$180,000, is closest to most of the values, it best represents the value of the seven cars.

This question asks which measure **best** represents the value of the seven cars. In other words, it is asking which measure best fits the cost of a race car. The answer to this question depends on how far the spread is.

In this case, \$819,000 is a value that drastically increases the mean. Also, more than half the cars fall around \$180,000.

<u>Definition</u>

- A **polynomial** is of the form

  $$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \qquad Where\ all\ a_i\ lie\ in\ \boldsymbol{F}\ and\ n \epsilon \boldsymbol{Z}^{\geq 0}.$$

  (For all **polynomials**, "n" is called the *degree of the polynomial*.)

- A **monomial** is a term in a **polynomial** of the form

  $$a_n x^n \qquad Where\ a_n \epsilon \boldsymbol{F}\ and\ n \epsilon \boldsymbol{Z}^{\geq 0}.$$

  (For all **polynomials**, "n" is called the *degree of the polynomial*.)

<u>Exercise</u> Using *2 variables*, list all possible **monomials** where the *sum of degrees is 3*.

$$x^3y^0,\ x^1y^2,\ \ x^2y^1,\ x^3y^3 \qquad Therefore, there\ are\ 4\ monomials$$

<u>Exercise</u> Using *k variables*, list all possible **monomials** where the *sum of degrees is n*. Let $k = 3, n = 2$.

$$x^2 \qquad xy$$

y^2 yz
z^2 xz

We have a total of *6 monomials*.

Exercise Let $k = 2, n = 3$. How many **monomials** can we list?

x^3	y^3	z^3
x^2y	xy^2	xz^2
x^2z	y^2z	yz^2
	xyz	

10 monomials

Question: Consider $v(k, n)$ where v gives us the number of monomials from *k variables* with the *sum of degrees n*. Is there a formula or method to find $v(k, n)$ $where\ k\epsilon\boldsymbol{N}, n\epsilon\boldsymbol{Z}^{\geq 0}$?

(Hint: Fix *k* to a value, then look for a pattern)

Fix *k*=*1*, then we have $v(1, n)$.

i) $v(1,0) = \{x^0 = 1\} = 1$
ii) $v(1,1) = \{x\} = 1$
iii) $v(1,2) = \{x^2\} = 1$
iv) $v(1,3) = \{x^3\} = 1$
v) $v(1,4) = \{x^4\} = 1$
vi) …keep repeating…

We can derive a general formula for $v(1, n) = \{x^n\} = 1 \quad \forall n\epsilon\boldsymbol{Z}^{\geq 0}$

Fix *k*=*2*, then we have $v(2, n)$.

i) $v(2,0) = \{x^0y^0 = 1\} = 1$
ii) $v(2,1) = \{x, y\} = 2$
iii) $v(2,2) = \{x^2, xy, y^2\} = 3$
iv) $v(2,3) = \{x^3, x^2y, xy^2, y^3\} = 4$

Suppose $v(2, n) = n + 1$. From the calculations above, this formula holds.

Let's take this a step further. **Fix *k*=*3***, then we have $v(3, n)$

i) $v(3,0) = \{x^0y^0z^0 = 1\} = 1$
ii) $v(3,1) = \{x, y, z\} = 3$
iii) $v(3,2) = \{x^2, y^2, z^2, xy, xz, yz\} = 6$
iv) $v(3,3) = \{x^3, y^3, z^3, x^2y, xy^2, xz^2, x^2z, y^2z, yz^2, xyz\} = 10$
v) …these calculations start to become tedious. However, we can still see a pattern.

Jeremy ferociously proposed that $v(3, n) = \sum_{i=0}^{n}(n + 1)$.

Then, Loren sprinted to the blackboard and wrote the following table…

n	$v(3, n)$
0	1

1	3
2	6
3	10
4	15

vi) Let's see whether our formula (or patterns) holds for $v(3,4)$.

$$v(3,4) = \{x^4, x^3y, x^3z,\ x^2y^2, x^2z^2, xy^3, xz^3, y^4, y^3z, y^2z^2, yz^3, z^4, x^2yz, xy^2z, xyz^2\} = 15$$

Loren's table is true. Now we can say that Jeremy's formula can be used.

Once more… **Fix *k*=*4***, then we have $v(4,n)$

i) $v(4,0) = \{x^0y^0z^0w^0 = 1\} = 1$

ii) $v(4,1) = \{x, y, z, w\} = 4$

iii) $v(4,2) = \{x^2, y^2, z^2, w^2, xy, xz, xw, yz, yw, zw\} = 10$

iv) $v(4,3) = \{x^3, x^2y, x^2z, x^2w, y^3, xy^2, y^2z, y^2w, z^3, xz^2, yz^2, z^2w, w^3, xw^2, yw^2, zw^2, xyz, xyw, xzw, yzw\} = 20$

Now suppose $v(4,n) = v(4,n-1) + v(3,n) \quad \forall n \geq 1$

Look at $v(4,3)$. According to our formula, $v(4,3) = v(4,2) + v(3,3) = 10 + 10 = 20$. The formula holds!

At this point, Keith, using his unearthly knowledge of Mathematics, pointed out a pattern! Another definition?!

<u>Definition</u>

- **Pascal's Triangle** is a infinite sequence of numbers listed in a triangular form below.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1
1 11 55 165 330 462 462 330 165 55 11 1
1 12 66 220 495 792 924 792 495 220 66 12 1
1 13 78 186 715 1287 1716 1716 1287 715 186 78 13 1

The entire border is made up of 1's. Every other number is the sum of the two numbers immediately above it.

Keith applied this triangle to the formula $v(k,n)$.

Keith's Method

i) Each row in Pascal's triangle applies to the corresponding *k* value.

ii) To find $v(k,n)$, go to the k^{th} row (**Notice** that the number next to the border is the corresponding row) and drop down and right (diagonally) *n* times.

iii) The number that you land on is the value of $v(k,n)$. (**Notice** that the same number is obtained from either side of the triangle. This is true since the triangle is symmetric.)

Ex 1 Find $v(7,4)$.

$$1 \to 7 \to 28 \to 84$$

Ex 1 Find $v(10,5)$.

$$1 \to 10 \to 55 \to 220 \to 715$$

Here's another method we can deduce from Keith's method!

1) From Keith's method, we can see that $v(k,0) = 1 \quad \forall k \epsilon \boldsymbol{N}$.
2) Now, we can convert the formula $v(4,n) = v(4,n-1) + v(3,n) \quad \forall n \geq 1$ into a general formula. Just replace 4 with *k*. Since $k = 4 \to 3 = k - 1$.

Now we have our formula!

Take the formula $v(k,n)$. When $n = 0$, $v(k,0) = 1 \quad \forall k \epsilon \boldsymbol{N}$.
Otherwise, $v(k,n) = v(k,n-1) + v(k-1,n) \quad \forall n \geq 1$.

References

Eccles, Peter J. (2007). *An Introduction to Mathematical Reasoning: numbers, sets and functions* (9th printing). New York: Cambridge University Press.

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