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Class Notes
October 7, 2008

Polynomials as Functions

Recognizing Algebraically/Graphically

Algebraically: General form of polynomials: $f(x) = a_0+a_1x+a_2x^2+ \ldots +a_nx^n$, n is a natural.

Ex. $g(x) = x^2+2x+10$

$h(x) = x^4+3x^3+2x$

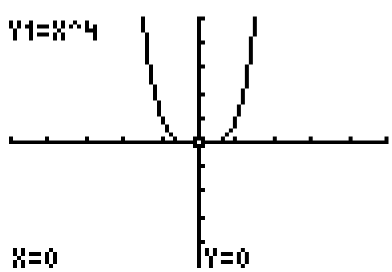
Graphically: Looking at the visual representation of a polynomial, the graph will be a curved function, as any straight line on the graph is y=mx+b or y=a, where any curved function will be at least degree 2, $y=x^2$.

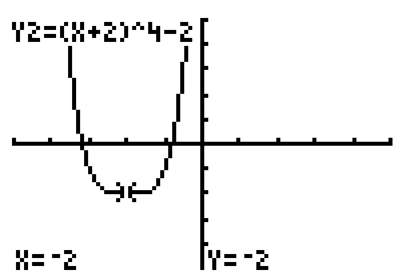
Shifting Polynomials on a graph

By examining $f(x) = (x)^n+a_0$, the graph can be shifted along the x-axis and y-axis. When $(x)^n$ has a constant, c, added to x, $(x+c)^n$, then it will shift the graph to the left when positive value, and to the right when negative, along the x-axis. This is because x+c acts as using a different value along the x-axis.

When a_0 is a non-zero integer, then the graph will shift up when a_0 is a positive value, and down when a negative value along the y-axis. This is because adding or subtracting a constant from the resulting value of the x^n will change the final value of f(x), as along the y-axis.

Ex. $f(x)=x^4$ $f(x)=(x+2)^4-2$





Inverses of polynomial functions

Not every polynomial function's inverse is a function.

Proof by contradiction:

For polynomial function, f(x), to have inverse $f^{-1}(x)$, then f(x) is injective and surjective for f(x) to have an inverse. If f(x) has a^n, with n>1, then f(x) is not necessarily injective.

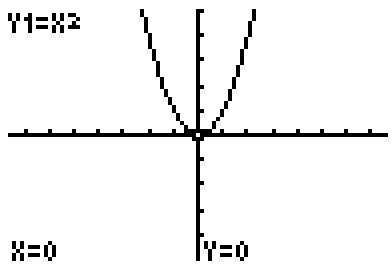
Counter example: $f(x)=x^2$

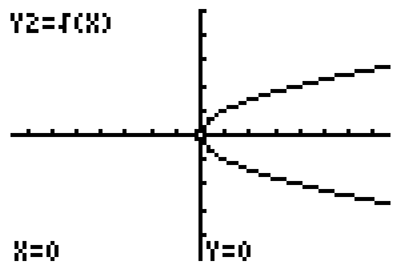
$f(x_1)=f(x_2) \Rightarrow x_1=x_2$ shows injectivity.

But since f(1)=f(-1) => 1=1, and 1 does not equal -1, then injectivity fails.

Therefore polynomial function f(x) does not have an inverse that is a function.

Ex. $p(x) = x^2$, is a function. $p^{-1}(x) = \sqrt{x}$, is not a function





By restricting the domain of p(x), then $p^{-1}(x)$ can be a function.

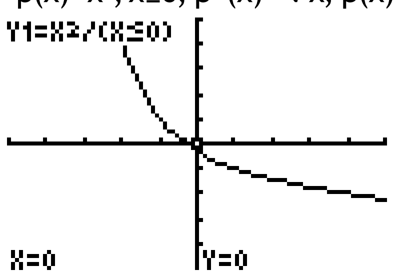
In order to look at an inverse function of a polynomial function, the graph must be broken into intervals that are each bijective. The end points of the intervals are found where the graph has a slope of zero, or equivalently, where the derivative of p(x), p'(x), is zero. The number of intervals will be less than or equal to the degree of the polynomial, depending on its behavior.

Ex. $p(x)=x^2 \Rightarrow p'(x)=2x=0 \Rightarrow x=0$

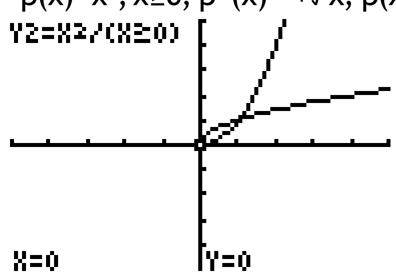
Intervals of p(x) will be $[-\infty,0]$ and $[0,+\infty]$

$p(x)=x^2, x\leq 0 \Rightarrow p^{-1}(x)= \sqrt{x}, x\leq 0$ and $p(x)=x^2, x\geq 0 \Rightarrow p^{-1}(x)= \sqrt{x}, x\geq 0$

$p(x)=x^2, x\leq 0, p^{-1}(x)=-\sqrt{x}, p(x)\leq 0$



$p(x)=x^2, x\geq 0, p^{-1}(x) =+\sqrt{x}, p(x)\geq 0$



Definition: Degree of Polynomials

Determined by the highest power of x^n in the function.

Ex. $f(x) = x^4+3x^3+2x$, has a degree of 4.

End behavior of Polynomial Functions

Even Degree: $f(x)=a_n x^n$, n=2k, k is an integer

With the leading coefficient, a_n, as a positive value:

Limit as x approaches ∞ → f(x) approaches ∞

Any even power of x will result in positive value of f(x) for x>0

Proof: Any positive multiplied by itself, will be a positive result, $(x)*(x)=+x^2$

All even powers of x can be broken into groups of $x^{2}*x^{2}*\ldots*\ x^{2}$, k times, and all x^2 are positive results.
Therefore, x^{2k} will be positive and approaching ∞.

Limit as x approaches -∞ → f(x) approaches ∞

Any even power of x will result in positive value of f(x) for x<0

Proof: Any negative multiplied with itself will be positive, $(-x)*(-x)=+x^2$

All even powers of x can be broken into groups of $x^{2}*x^{2}*\ldots*\ x^{2}$, k times, and all x^2 are positive results.
Therefore, x^{2k} will be positive and approaching ∞.

By both of these results, each end point of an even degree polynomial function will be approaching infinity.

If the leading coefficient, a, is negative, then the opposite result will be true, f(x) approaching negative infinity on both ends.

Jeremy's thought: Polynomial functions of even degree are not invertible.
By looking at the end behavior of even polynomial functions, then all share the same behavior as $p(x)=x^2$. As shown previous proof, x^2 is not bijective, therefore x^2 is not invertible. Then all functions with similar end behavior will not be invertible. Then all even polynomial functions are not invertible.

Odd Degree: $f(x)=a_nx^n$, n=2k+1, k is an integer

With the leading coefficient, an, as a positive value:

Limit as x approaches ∞ → f(x) approaches ∞

Any odd power of x will result in positive value of f(x) for x>0

Proof: Any positive multiplied by itself, will be a positive result,
$(x)*(x)=+x^2$
All odd powers of x can be broken into groups of $(x^{2}*x^{2}*\ldots*\ x^{2})*x$, with x^2 k times, and all x^2 are positive results, times x, another positive, will be a positive result.
Therefore, x^{2k+1} will be positive and approaching ∞.

Limit as x approaches -∞ → f(x) approaches -∞

Any odd power of x will result in positive value of f(x) for x>0

Proof: Any negative multiplied by itself, will be a positive result,
$(-x)*(x)=+x^2$

All odd powers of x can be broken into groups of $(x^{2}*x^{2}* \ldots * x^{2})*x$, with x^2 k times, and all x^2 are positive results, times x, a negative, will be a negative result.
Therefore, x^{2k+1} will be negative and approaching -∞.

By both cases, the end behavior of an odd degree polynomial function will have opposite behavior, with the left side approaching negative infinity and the right side approaching positive infinity.

If the leading coefficient, a_n, is negative, then the opposite result will be true, f(x) approaching positive infinity on the left, and negative infinity on the right.

The Fundamental Theorem of Algebra: Every polynomial p(x) of degree n>0 has at least one zero in the complex number system.
Definition: Roots of Polynomial Functions
The roots are where $f(x)=0=a_0+a_1x+a_2x^2+\ldots+a_nx^n$.
f(x) will have a number of roots equal to its degree.
Types of Roots
1) Where f(x) crosses the x-axis, y=0.
   Ex. $f(x)=0=x^2-4 \Rightarrow x=\pm 2$, so $f(\pm 2)=0$.
2) Where f(x) touches the x-axis.
   Ex. $f(x)=0=x^2 \Rightarrow x*x=0 \Rightarrow x=0$, mult 2, so f(0)=0.
3) Complex Roots: Any zeros of f(x) that contain a negative square and in the form a+b*i*.
   Ex. $f(x)=0=x^2+x+2 \Rightarrow x=[-1\pm\sqrt{(1^2-4(1)2)}]/2 \Rightarrow x=[2\pm\sqrt{(-4)}]/2 \Rightarrow x=1\pm i$, so $f(1\pm i)=0$.

Even Degree Polynomial Roots
Will have an even number of roots, and these roots will be in sets of distinct roots (1), mult n, n=2k (2), or complex (a+b*i*, a-b*i*) (3) that the number of roots add to the degree.
Ex. Parabola, $y=a_2x^2+a_1x+a_0$
$f(x)=x(x-1)$, 2 distinct roots, 2 roots in all.
$g(x)=(x-1)^2$, 1 root mult 2, 2 roots in all.
$h(x)=x^2+x+2$, 2 complex roots, 2 roots in all.

Odd Degree Polynomial Roots
Will have an odd number of roots, and these roots will be in sets of distinct roots, an odd number (1), mult n, (2), or sets of complex (a+b*i*, a-b*i*) (3) that the number of roots add to the degree.
Ex. Cubics, $y=a_3x^3+a_2x^2+a_1x+a_0$

$f(x)=x(x-1)(x+2)$, 3 distinct roots, 3 roots in all.
$g(x)=x(x-1)^2$, 1 distinct root, 1 root mult 2, 3 roots in all.
$h(x)=(x-1)^3$, 1 root, mult 3, 3 roots in all.
$j(x)=x^3-1$, 1 distinct root, 2 complex roots, 3 roots in all.

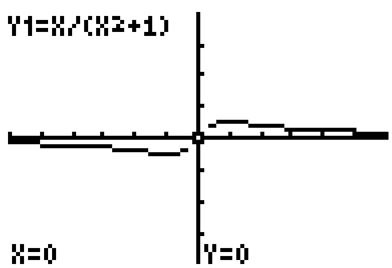
Types of Common Polynomials

Linear- Degree 1 polynomial
$y(x) = mx+b$
Quadratic- Degree 2 polynomial
$q(x) = ax^2+bx+c$
Cubics- Degree 3 polynomial
$c(x) = ax^3+bx^2+cx+d$

Asymptotes in polynomial functions

Horizontal Asymptotes are values of y=a, such that for x approaching $\pm\infty$, $f(x)\neq a$. So f(x)=y, for some x,
Limit as x approaches ∞ → f(x) approaches a
or Limit as x approaches $-\infty$ → f(x) approaches a
This means as x gets larger, the value of y get arbitrarily closer to a.

Ex. $f(x)=1/(x^2+1)$, f(x) approaches 0 as x approaches ∞ and $-\infty$.



Vertical Asymptotes are values of x=a, such that for all x, $f(x)\neq a$. So for some x,
Limit as f(x) approaches $a^{\pm}$ → f(x) approaches ∞
or Limit as f(x) approaches $a^{\pm}$ → f(x) approaches $-\infty$
This means as f(x) approaches a, f(x) goes to ∞ or $-\infty$.

Ex. $f(x)=1/(x+1)$, $f(x)\neq -1$,
Limit as f(x) approaches a^- → f(x) approaches $-\infty$
or Limit as f(x) approaches a^+ → f(x) approaches ∞