MAE 301/501: FOUNDATIONS OF SECONDARY SCHOOL MATHEMATICS

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Contents

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1. Review of Homework #2

At the start of the day's discussion, we began by looking to the previous graded homework, which was returned during this class meeting. If you recall, this assignment concerned functions whose domain and codomain are the integers. Moreover, the assignment was concerned with giving examples; examples of functions $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ that were injective, surjective, bijective, or neither.

Here, let us recall the definitions of injective, surjective, and bijective. We begin with the definition of injective.

Def. Let f be a function with domain D. The function f is injective if for each $x, y \in D$, if $f(x) = f(y)$, then $x = y$.

We follow with the definition of surjective.

Def. A function $f : X \longrightarrow Y$ is surjective if and only if for each y in the codomain Y, there is at least one x in the domain X such that $f(x) = y$.

Finally, with the definitions of injective and surjective, it is very easy to say what we mean when we say a function is bijective.

Def. A function $f : X \longrightarrow Y$ is bijective if it is both injective and surjective.

Returning to our main focus, we wanted to give some exposition to at least one problem from the homework. The problem asked to find an example of a function $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ that was bijective. Our colleague, Anne, proposed a solution from her homework assignment. Her suggestion was to consider the function $f(x) = x + 1$. Her proof that $f(x)$ is bijective follows below.

Proof. For Anne, she begins by demonstrating that $f(x) = x + 1$ is injective. Similar to the above stated definition, Anne's criteria is that she will show if $f(x_1) = f(x_2)$, then it will imply that $x_1 = x_2$. We begin by assuming for arbitrary values $x_1, x_2 \in \mathbb{Z}$, that $f(x_1) = f(x_2)$. Applying this to $f(x)$ gives:

$$
\begin{aligned}
x_1 + 1 &= x_2 + 1 \\
x_1 + 1 - 1 &= x_2 + 1 - 1 \\
x_1 &= x_2
\end{aligned}
$$

Since we are able to yield that $x_1 = x_2$, then it must be that $f(x) = x+1$ is injective.

It now remains to show that $f(x) = x+1$ is surjective. Again, similar to the afore mentioned definition, the criteria Anne chose to consider was that a function is surjective if for each $y \in \mathbb{Z}$, there is an $x \in \mathbb{Z}$ so that one has $f(x) = y$.

We begin by considering any $y \in \mathbb{Z}$. Then, by the definition of surjective, we are looking for when $f(x) = y$. Thus, if $y = x + 1$, then $x = y - 1$. So now, set $x = y - 1$, then:

$$\begin{aligned} f(x) &= x + 1 \\ &= (y - 1) + 1 \\ &= y - 1 + 1 \\ &= y \end{aligned}$$

Thus, indeed we have the condition $f(x) = y$ as desired and it follows that $f(x)$ is surjective. This completes the proof that $f(x) = x + 1$ is bijective.

□

Following the proof given by Anne of the claim that $f(x) = x + 1$ is a bijective function from $\mathbb{Z}$ to $\mathbb{Z}$, a comment had been made suggesting that the function $y = \tan x$ was an example of a function with domain $\mathbb{Z}$ and codomain $\mathbb{Z}$ that was surjective.

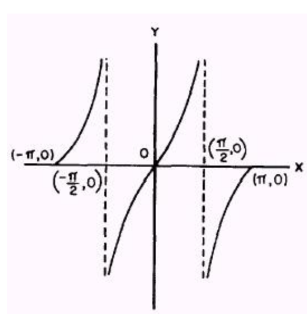

FIGURE 1. Graph of $y = \tan x$

However, as quickly as the function was suggested as one that was surjection from integers to integers, it was also quickly retracted as it was pointed out that if we suppose $y = 1$, then for value or values of x does $\tan x = 1$. Clearly, we saw from the graph that already at $x = \frac{\pi}{4}$ and $\frac{3\pi}{4}$ the graph took value $y = 1$. That is, $\frac{\pi}{4}$ and $\frac{3\pi}{4}$ are not in $\mathbb{Z}$, and thus, not in the domain. Hence, nothing in the domain maps to $y = 1$.

Finally, a question was posed to our as what is the difference between codomain and range. Informally, codomain was defined to be what you are aiming for, the so-named "target space," while, the range was defined to be what you hit. Somewhat more formally, we may define the two terms as follows.

Def. The codomain or target of a function is the set Y into which all of the output of the function is constrained to fall. It is the set Y in the notation, $f : X \longrightarrow Y$.

While, we may characterize the range in the following way.

Def. The image or range of a function is the image of the domain of the function in the codomain. If the function is a surjection, then the image is equal to the codmain.

Resulting from our discussion regarding this clarification, it was asserted that one may think of the following inclusion

$$\text{Range} \subset \text{Codomain}.$$

2. Rational Functions

Once we were satisfied with a discussion of some of the homework problems, as well as ideas related to the homework, we were able to move onto our primary objective for the day's lecture which was to continue our discussion of rational functions.

We recall that during our previous lecture, on September 18, 2009, we had already thought about rational functions and given a preliminary definition. That is, based on how we saw polynomials as equivalent, we then abstracted rational functions to be equivalence classes of ratios of polynomials. That is, we thought of the example of

$$\frac{1}{x} \sim \frac{x-1}{x^2-1}$$

as equivalent when viewed as ratios of polynomials.

However, lets make our notion of rational function more precise. We offer the following definition.

Def. The set of rational functions is the set of all ratios of the form

$$\{\frac{p}{q} \mid p, q \in K[x],\ \ q \neq 0 \text{ where } \frac{p}{q} \sim \frac{p'}{q'} \Longleftrightarrow pq' = qp'\}.$$

Note that $K[x]$ denotes an arbitrary polynomial ring in one indeterminate and $q \neq 0$ is meant to mean that q is not the zero polynomial and not to be confused with some value x_0 in a domain D such $q(x_0) = 0$.

In addition, notice, this definition provides a good notion of algebraic equivalence. That is, in the example above, we can say as algebraic expressions the two separate expressions are equivalent. For, since $x(x-1) = x^2 - x$, then we are justified to say equivalent. However, we when viewed as functions, we cannot say that they are equal, we may only say that one is an equivalent form of the original function.

From this reasoning, we considered two (generally) equivalent definitions characterizing when we may say rational functions are equivalent as functions.

Def. (1) Two rational functions are equivalent if, as functions, they are equal on all shared points of their domains.

$$\Updownarrow$$

Def. (2) Two rational functions are said to be equivalent algebraically if when one restricts to the intersection of their domains, then they become equal as functions.

Following a direct comparison of both definitions, we returned to our example of the rational function $f(x) = \frac{x-1}{x^2-x}$. Again, to state our point of "equivalent as functions," we review the computation, i.e.,

$$\frac{x-1}{x^2-x} = \frac{x-1}{x(x-1)} = \frac{1}{x} \ \ x \neq 1.$$

In the case of this example, it was noted that some teachers will either mention that point that $x \neq 1$ in their definitions or they will not. This became something to consider in terms of what does one say for algebraic equivalence versus equivalent as functions.

2.1. **Asymptotes.** Following our discussion of rational functions, a question was posed to as when is the graphical behavior of a rational function effected by its form? That is, what do you call the parts of the rational function that effectively give denominator zero from its domain or otherwise? We answered this question by considering the notion of an asmyptote. More precisely, we identified that rational functions could have two types of asymptotes: a vertical asymptote and a horizontal asymptote.

Note, do not let the grammatic construction mislead you to believe that rational functions have only one vertical asymptote and one horizontal asymptote. In fact many rational functions have many vertical asymptotes. However, for horiztonal asymptotes there is only one associated to a rational function. Consider examples such as

(a) $f(x) = \dfrac{x^2 + 4x + 3}{x^4 - 4x^2 + 3}$
(b) $g(x) = \arctan x$

For this part of the class, the task was presented to the students to agree upon reasonable definitions of vertical and horizontal asymptotes. Here, our colleagues, James and Nick offered some insight.

Def. (Nick) A line at which a value for x such that the function approaches that value but never touches it.

Comparatively, versus our other colleague James, who makes the following definition.

Def. (James) Let $f(x)$ be a a function that approaches a value t such that the limit gets infinitely close to t but never obtains that value.

Upon immediate acceptance of both definitions, our next assignment was to test the validity or accuracy of both conjectured definitions. Again, we looked to our prevuous example, $y = \tan x$. Upon examination of the graph of the function, we noted that at $x = \frac{\pi}{2}$ and values of the form $x = \frac{\pi}{2} + k\pi$ for $k \in \mathbb{Z}$, the graph of tangent had vertical asymptotes. Moreover, invoking the notion of a certain x-value approaching a number but only ever being able to be infinitely close seemed feasible with this particular graph.

One critical point of disagreement was reached while forming the afore mentioned observation. That is, during the discussion of the definition of vertical asymptote, a broad confusion developed regarding the technical nature of what was approaching what. For many, a statement was made that, "this is approaching this" and it is "infinitely" close. However, what was soon revealed was that many people seemed to be saying the following incorrect statement: "the function approaches this." We immediately corrected this by saying some x in the domain is approach a quantity on the x-axis, and as this number is getting closer to x, the function tends to "this" behavior.

We remedied our confusion by again returning to $y = \tan x$. It was made clear that the following true about $\tan x$. That as x approached $\frac{\pi}{2}$, the function $\tan x$ was approaching $+\infty$. This showed us that in fact $\tan x$ approached $+\infty$ as x approaches $\frac{\pi}{2}$.

At this juncture, we had exhausted the definition of vertical asymptote in a elementary setting and were ready to modify our definition for our target audience of high students perhaps in Pre-Calculus or Calculus. Again, our colleague James offered his insights into a possible definition.

Def. (James) A rational function f has a vertical asymptote at $x = a$ if $\lim\limits_{x \longrightarrow a^-} f = \pm\infty$ or $\lim\limits_{x \longrightarrow a^+} f = \pm\infty$.

This definition, although, generally, complete did not satisfy the audience. Several similar questions were posed. To some extent they centered on the following idea: Is the stated definition true? i.e., couldn't there be some situation that violates the definition? That is, others wanted to replace "approaching ∞" with "has no limit." Two examples were offered to counter this replacement claim.

(a) Consider the function $y = \sin x$ on the interval $0 \leqslant x \leqslant 2\pi$. As $x \longrightarrow +\infty$, $\sin x \longrightarrow +\infty$. This example was evident.
(b) Consider a sequence with the representation

$$\{a_n\}_{n=1}^{\infty} = (-1)^n = \begin{cases} 1 & n \text{ even} \\ -1 & n \text{ odd} \end{cases}.$$

> This sequence when viewed as a function continuously oscillates between -1 and 1. There is, for example, no well-defined limit as $x \longrightarrow 1$. Note however from analysis, we know that $\limsup\{a_n\} = 1$ and $\liminf\{a_n\} = -1$. Hence, there is no limit.

Our final task for the day was to consider a reasoning task. To formulate two versions of a definition of vertical asymptote. One for an audience with enough mathematical foundation to have a fair understanding, while another for an audience who is perhaps enrolled in a course on the prinicples of analysis.

It was generally agreed on the following definition fit the requirements for persons with no exposure to analysis.

Def. The function f has a vertical asymptote at $x = a$ if as x approaches a from the right hand-side, the function goes to $\pm\infty$. Similarly, if x approaches a from the left hand-side, the function goes to $\pm\infty$.

It is worthy to mention that as future teachers, part of our job will be to bring clarity with one avenue being a good definition. The above definition grew out the following two earlier versions.

Version A. $y = f(x)$ where 1 or more real numbers equal to some x such that the limit of real numbers approaches $\pm\infty$ in the y-direction.

Version B. If $y = f(x)$ is a function that approaches $\pm\infty$ in the y-direction as x approaches 1 or more values, then there is a vertical asymptote at $x = a$.

At this point, an objection was made to the construction of the definition. That is, a student felt that perhaps something made the definition incomplete. In brief, the argument came down to addressing the question of: provide an example of a function that is undefined at a point, but has no asymptote at that point. We considered the following function.

$$f(x) = \frac{x-1}{x^2-1} = \frac{x-1}{(x-1)(x+1)} = \frac{(x \not- 1)}{(x \not- 1)(x+1)} = \frac{1}{x+1}.$$

Following the example, the student understood what made the function undefined. When $x = 1$, the function has denominator zero. In addition, the student pointed out that the resulting equivalent function was undefined at $x = -1$. A question was posed to the student, what do you call the point $x = 1$ for which the function is undefined. The answer: a pinhole, a removable discontinuity, etc.

Lastly, we agreed upon a definition of vertical asymptote we felt a student in analysis or having taken analysis could be aware of.

Def. A function $f(x)$ has a vertical asymptote if, for every real number M, there is a positive number δ so that whenever the distance between x and a is small, i.e., $|x - a| < \delta$, then this implies that $f(x) > M$.

3. Problems

Here are some problems for the reader to consider at a later time.

1. Without appealing to Calculus, graph the rational function

$$f(x) = \frac{x^2 - 4x + 3}{x^4 - 17x^2 + 16}.$$

2. Find the limit: $\lim\limits_{x \longrightarrow 1} \frac{|x-1|}{x-1}$.

3. Let $f : \mathbb{R} \longrightarrow \mathbb{R}^+$ be a strictly monotone, increasing function (possibly a convex function) and let $a \in \mathbb{R}$. Find, with proof, $\lim\limits_{x \longrightarrow a} f(x)$.

4. Let λ be any positive real number and let $f(x)$ and $g(x)$ be two rational functions. From our definition of rational functions as equivalence classes of ratios of polynomials, can we say that $f \sim g$ if $f(x) = g(\lambda x)$? Justify your reasoning.