SOLUTIONS FOR HOMEWORK SET 11

4.2.4.

a: boundary point, hence not local extremum, not global extremum either
b,c,d: local minimum, not a global minimum
e: global maximum and local maximum
r: local minimum
s: local maximum
t: global minimum, but not local minimum

4.2.6.

absolute minimum: $f(1) = 0$
absolute maximum: $f(7) = 5$
local minima: $f(1) = 0, f(4) = 2, f(6) = 1$
local maxima: $f(0) = 2, f(3) = 4, f(5) = 3$

4.2.10. See the picture at the end.

4.2.24.

$$\begin{aligned} f'(x) &= 3x^2 + 2x - 1 \\ 3x^2 + 2x - 1 &= 0 \\ x_{1,2} &= -\frac{1}{3} \pm \sqrt{\frac{1}{9} + \frac{1}{3}} \\ &= -\frac{1}{3} \pm \sqrt{\frac{4}{9}} \\ &= -\frac{1}{3} \pm \frac{2}{3} \end{aligned}$$

The critical points of f are $x = \frac{1}{3}$ and $x = -1$.

4.2.32.

$$\begin{aligned} g'(\theta) &= 1 + \cos\theta \\ 1 + \cos\theta &= 0 \\ \cos\theta &= -1. \end{aligned}$$

The critical points of g are $\theta = (2k+1)\pi$ for any integer k.

4.2.36. (1) Find all critical points.

$$\begin{aligned} f'(x) &= 3x^2 - 3 \\ 3x^2 - 3 &= 0 \\ x &= \pm 1. \end{aligned}$$

The only critical point in $[0,3]$ is $x = 1$.
(2) Evaluate f at its critical points and the boundary of the interval $[0,3]$ and determine the minimum and the maximum among these values:

$$\begin{aligned} f(0) &= 1 \\ f(1) &= -1 \\ f(3) &= 19. \end{aligned}$$

The maximum of f is $f(3) = 19$ its minimum $f(1) = -1$.

4.2.42. Same procedure as in previous problem:

$$\begin{aligned} f'(x) &= 1 + 2\sin x \\ 1 + 2\sin(x) &= 0 \\ \sin(x) &= \frac{1}{2}. \end{aligned}$$

The critical points of f in $[-\pi, \pi]$ are $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$.

$$\begin{aligned} f(-\pi) &= -\pi + 2 \\ f(\frac{\pi}{6}) &= \frac{\pi}{6} - \sqrt{3} \\ f(\frac{5\pi}{6}) &= \frac{5\pi}{6} + \sqrt{3} \\ f(\pi) &= \pi + 2. \end{aligned}$$

The maximum of f is $f(\pi) = \pi + 2$, its minimum is $f(-\pi) = -\pi + 2$.

4.2.52. Same procedure as in previous problems:

$$\begin{aligned} F'(\theta) &= -\frac{(\mu\cos\theta - \sin\theta)\mu W}{(\mu\sin\theta + \cos\theta)^2} \\ (\mu\cos\theta - \sin\theta) &= 0 \end{aligned}$$

The critical point in $[0, \pi/2]$ satisfies $\tan\theta = \mu$. Remains to check that it is minimizing!! $\tan\theta = \frac{\sqrt{1-\cos^2\theta}}{\cos\theta}$. Therefore $\mu^2\cos^2\theta = 1 - \cos^2\theta$ and $\cos\theta = \frac{1}{\sqrt{1+\mu^2}}$

$$\begin{aligned} F(0) &= \mu W \\ F(\frac{\pi}{2}) &= W \\ F(\theta) &= \frac{\mu W}{(\mu^2+1)\cos\theta} \\ &= \frac{\mu W}{\sqrt{\mu^2+1}}. \end{aligned}$$

The last value is smaller than any of the previous two. □

4.3.2. (a) The (largest) intervals where g is concave upward are $(-1, 2)$, and $(7, 8)$.
(b) The (largest) intervals where g is concave downward are $(2, 4)$, and $(4, 7)$.
(c) The coordinates of the inflection point are $(2; 2)$.

4.3.6. (a) f is increasing on $[2, 4]$, and $[6, 9]$ since $f' \geq 0$ on these intervals.
(b) f has a local maximum at $x = 4$ since $f'(4) = 0$ and $f'(x)$ changes sign from positive to negative at that point, local minima at $x = 2$ and $x = 6$ since $f'(2) = f'(6) = 0$ and $f'(x)$ changes sign from negative to positive at these points.
(c) f is concave upward on $(1, 3)$, $(5, 7)$, and $8, 9)$ since f' is increasing on these intervals, it is concave downward on $(0, 1)$, $(3, 5)$, and $(7, 8)$ since f' is decreasing on these intervals.
(d) The x coordinates of the inflection points are $x = 1$, $x = 3$, $x = 5$, $x = 7$, and $x = 8$. At these points $f''(x)$ vanishes.

4.3.16.

$$\begin{aligned} f'(x) &= 4x^3(x-1)^3 + 3x^4(x-1)^2 \\ &= x^3(x-1)^2(4(x-1)+3x) \\ &= x^3(x-1)^2(7x-4) \\ f''(x) &= 3x^2(x-1)^2(7x-4) + 2x^3(x-1)(7x-4) + 7x^3(x-1)^2 \\ &= x^2(x-1)(3(x-1)(7x-4) + 2x(7x-4) + 7x(x-1)) \\ &= x^2(x-1)(42x^2 - 48x + 12). \end{aligned}$$

(a) The critical numbers are $x = 0$, $x = 1$, and $x = \frac{4}{7}$.
(b) $f''(0) = f''(1) = 0$. Hence we cannot conclude that either of these points is a local extremum–at least not from the Second Derivative Test. On the other hand, $f''(\frac{4}{7}) = (\frac{4}{7})^2(-\frac{3}{7})(4(-\frac{3}{7})) > 0$. Hence $x = \frac{4}{7}$ is a local minimum.
(c) For $x < 0$ and close to 0 $f'(0)$ is positive while it is negative for $x > 0$ and close to it:

Notice that $(0-1)^2(7\times 0-4)<0$ and this remains true for x close to 0!! Hence f has a local maximum at $x=0$. For either $x<1$ and $x>1$ and close to 1 the sign of $f'(x)$ is determined by $1^3(7\times 1-4)>0$. Hence $f'(x)$ is positive before and after $x=1$ and this is not a local extremal point. The conclusion for $x=\frac{4}{7}$ has to be the same as under (b)!

4.3.18.

$$\begin{aligned} g'(x) &= 24x^2+4x^3 \\ &= 4x^2(6+x) \\ g''(x) &= 48x+12x^2 \\ &= 12x(4+x). \end{aligned}$$

(b) The critical points are $x=0$ and $x=-6$. $g''(-6)=12(-6)(-2)>0$, hence g has a local minimum at this point. $g''(0)=0$! But $g'(x)>0$ on either side of $x=0$ and hence this is not a local extremal point.

(a) g is decreasing on $(-\infty,-6]$ and increasing on $[-6,+\infty)$ since $g'(x)$ is negative on the frist and positive on the second interval.

(c) $g''(x)=0$ for $x=0$ and $x=-4$. The function g is concave up on $(-\infty,-4]$ and concave down on $[-4,0]$ and concave up again on $[0,\infty)$.

(d) See the pictures at the end.

4.3.26. (a) The function $f(x)$ has a pole (vertical asymptote) at $x=1$.

$$\lim_{x\to 1^\pm}\frac{x}{(x-1)^2}=\lim_{x\to 1^\pm}\lim_{x\to 1^\pm} x\frac{1}{(x-1)^2}=1\times\lim_{x\to 1^\pm}\frac{1}{(x-1)^2}=+\infty.$$

Moreover,

$$\lim_{x\to\pm\infty}\frac{x}{(x-1)^2}=0.$$

hence $y=0$ is the (only) horizontal asymptote of f.

(b) We compute

$$f'(x)=\frac{(x-1)^2-2x(x-1)}{(x-1)^4}=\frac{-x^2+1}{(x-1)^4}=\frac{-x-1}{(x-1)^3}.$$

$f'(x)>0$ if $x^2<1$. Hence f is increasing on $[-1,1)$ and decreasing outside that interval on $(-\infty,-1]$ and $(1,\infty)$.

(c) From (b) follows that $x=-1$ is a local minimum. In fact from the horizontal asymptotes follows that this a global minimum.

(d) We compute

$$f''(x)=\frac{-(x-1)^3-(-x-1)3(x-1)^2}{(x-1)^6}=\frac{-(x-1)-3(-x-1)}{(x-1)^4}=\frac{2x+4}{(x-1)^4}.$$

$f''(x) = 0$ if $x = -2$. Hence the inflection point is at $x = -2$. The function is concave up on $[-2, 1)$ and $(1, \infty)$ and concave down on $(-\infty, -2]$.
(e) See the pictures at the end.

4.3.38. See the pictures at the end.
The graph says that there is one point in between the two (blue) particles where the net force vanishes and that the red particle is drawn to the blue particle to which it is closest if its position is different from that position.

4.3.42.

$$\begin{aligned} f'(x) &= ae^{bx^2} + ax(2bx)e^{bx^2} \\ &= a(1 + 2bx^2)e^{bx^2}. \end{aligned}$$

Hence $f'(2) = 0$ if $0 = 1 + 2b2^2 = 1 + 8b^2$ or $b = -\frac{1}{8}$. Now $1 = f(2) = 2ae^{-\frac{2^2}{8}} = 2a(\sqrt{e})^{-1}$ if $a = \frac{\sqrt{e}}{2}$.

4.3.46. Notice that f is differentiable by assumption. By the mean value theorem there exists a value $x \in [2, 5]$ such that

$$f(5) - f(2) = (5 - 3)f'(x) = 3f'(x).$$

Multiplying $1 \leq f'(x) \leq 4$ by 3 we obtain $3 \leq 3f'(x) \leq 12$ and hence $3 \leq f(5) - f(2) \leq 12$.

4.5.2. (a) is of the form "$0 \times \infty$". (b) and c are both equal to ∞.

4.5.6. Notice that $b \neq 0$ necessary!!

$$\lim_{x \to 1} \frac{x^a - 1}{x^b - 1} = \lim_{x \to 1} \frac{ax^{a-1}}{bx^{b-1}} = \frac{a}{b}.$$

4.5.12.

$$\lim_{x \to \infty} \frac{\ln \ln x}{x} = \lim_{x \to \infty} \frac{\frac{1}{\ln x}\frac{1}{x}}{1} = \lim_{x \to \infty} \frac{\frac{1}{x}}{\ln x} = \lim_{x \to \infty} \frac{-\frac{1}{x^2}}{\frac{1}{x}} = \lim_{x \to \infty} -\frac{1}{x} = 0.$$

4.5.30.

$$\lim_{x \to 1}\left(\frac{1}{\ln x} - \frac{1}{x - 1}\right) = \lim_{x \to 1} \frac{(x - 1) - \ln x}{\ln x (x - 1)} = \lim_{x \to 1} \frac{1 - \frac{1}{x}}{\ln x + \frac{x-1}{x}} =$$

$$\lim_{x \to 1} \frac{x - 1}{x \ln x + (x - 1)} = \lim_{x \to 1} \frac{1}{x\frac{1}{x} + \ln x + 1} = \frac{1}{2}$$

4.5.34. Notice that if $b = 0$ then the expression is identically equal to 1. Suppose therefore that $b \neq 0$.

$$\ln((1+\frac{a}{x})^{bx}) = bx\ln(1+\frac{a}{x}) = \frac{\ln(1+\frac{a}{x})}{\frac{1}{bx}}.$$

Now

$$\lim_{x\to\infty}\frac{\ln(1+\frac{a}{x})}{\frac{1}{bx}} = \lim_{x\to\infty}\frac{(1+\frac{a}{x})^{-1}(-\frac{a}{x^2})}{-\frac{1}{bx^2}} = \lim_{x\to\infty}\frac{(1+\frac{a}{x})^{-1}(-a)}{-b} = \frac{a}{b}.$$

Hence

$$\lim_{x\to\infty}(1+\frac{a}{x})^{bx} = e^{\frac{a}{b}}.$$