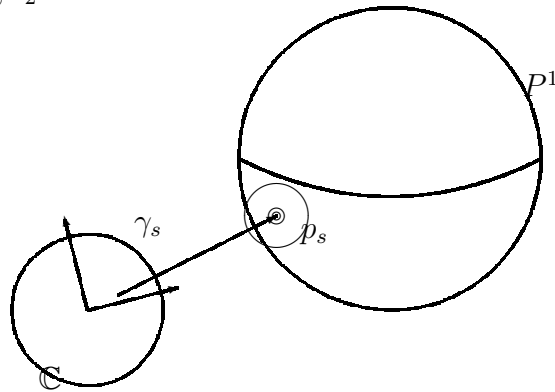


# Exercises on coinvariant bundles (work in progress)

Yau Kwan Kiu

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We continue the fusion exercise in `fusionex.tex` and study the Kazhdan-Lusztig fusion product  $\otimes$  in the simple case over the Riemann sphere with three points  $s_0, s_1, s_2$  removed.



## 1 The parameter space

When  $s_0 = \infty, s_1 = 0, s_2 = 1$ , we can study the modules over the Lie algebra  $\mathfrak{g}(\mathbb{C}[1/t, 1/(1-t)])$  with coefficients regular functions over  $\mathbb{P}^1 - 0, 1$ .

We now allow the points  $s_0, s_1, s_2$  to move. Consider a map  $s : \mathbb{P}^1 \rightarrow \mathbb{P}^1 : s(s_1) = 0, s(s_2) = 1$ . We then have the Lie algebra  $\mathfrak{g}(\mathbb{C}[\mathbb{P}^1 - \{s_1, s_2\}]) =$

$\mathfrak{g}(\mathbb{C}[1/s, 1/(1-s)])$ .

Following Kazhdan-Lusztig II, §7, we let  $H = PGL_2(\mathbb{C})$  be the group of morphisms of the projective line  $\mathbb{P}^1(\mathbb{C})$ .

As before, let  $S = \{s_0, s_1, s_2\}$ ,  $\heartsuit = \{s_0\}$ ,  $\spadesuit = \{s_1, s_2\}$ ,  $C = \mathbb{P}^1 - \{s_0, s_1, s_2\}$ ; Let  $V_1, V_2$  be highest weight modules (e.g. Weyl modules  $V_{\lambda_1}, V_{\lambda_2}$  over  $\tilde{\mathfrak{g}}$  generated by irreducible  $\mathfrak{g}$ -modules  $L_{\lambda_i}$ ).

Over on  $\mathbb{P}^1$  we consider the parameter space  $\mathcal{V} = \{(\gamma_1, \gamma_2) | \gamma_1(0) \neq \gamma_2(0)\}$  of two charts  $\gamma_1, \gamma_2 : (\mathbb{P}^1 \rightarrow \mathbb{P}^1) \in H$  at distinct positions on the projective line, and the parameter space  $\mathcal{V}' = \{(\gamma_1, \gamma_2, z) | \gamma_1(0), \gamma_2(0), z \text{ all distinct} \in \mathbb{P}^1\}$  of two charts plus a point  $z \in \mathbb{P}^1$  at distinct positions on the projective line.

## 1.1 Spaces of functions over the parameter spaces

Now let us calculate the spaces of functions  $\mathbb{C}[\mathcal{V}], \mathbb{C}[\mathcal{V}'], \mathbb{C}[\underline{\mathcal{V}}], \mathbb{C}[\underline{\mathcal{V}}']$ .

First of all, we have

$$\mathbb{C}[H] = \mathbb{C}[PGL_2(\mathbb{C})] = \mathbb{C}[a, b, c, d, \frac{1}{ad-bc}] |_{f(ka, kb, kc, kd)=f(a,b,c,d)}$$

where  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H$  i.e. a regular function on  $H$  is a homogeneous polynomial in  $a, b, c, d, 1/(ad-bc)$  with degree zero.

Next we have

$$\mathbb{C}[H \times H] = \mathbb{C}[a, b, c, d, a', b', c', d', \frac{1}{ad-bc}, \frac{1}{a'd'-b'c'}] |_{f(ka, kb, kc, kd, ma', mb', mc', md')=f(a,b,c,d,a',b',c',d')}$$

We have  $\mathcal{V} \subset H$ . The condition for a pair of charts  $\gamma_1, \gamma_2$  to be in  $\mathcal{V}$  is that their images at zero are distinct, i.e.  $b/d \neq b'/d'$ , or  $b'd - d'b \neq 0$ , where  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \gamma_1, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \gamma_2$ . So we have

$$\tilde{A} = \mathbb{C}[\mathcal{V}] = \mathbb{C}[a, b, c, d, a', b', c', d', \frac{1}{ad-bc}, \frac{1}{a'd'-b'c'}, \frac{1}{b'd-d'b}]$$

To calculate  $A = \mathbb{C}[\underline{\mathcal{V}}]$ , we may take the subring of  $H$ -invariants of  $\tilde{A}$  calculated above. Alternatively, consider

$$\underline{\mathcal{V}} \simeq H \setminus \{(\gamma_0, \gamma_1) | \gamma_0(0) \neq \gamma_1(0)\} \simeq \{(1, \gamma'_1 = \gamma_0^{-1}\gamma_1) | \gamma'_1(0) \neq 0 \in H\}$$

$$\simeq \left\{ \gamma'_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H \mid \frac{b}{d} \neq \frac{0}{1} \right\}$$

Then  $\underline{\mathcal{Y}} \subset H$  and so

$$A = \mathbb{C}[\underline{\mathcal{Y}}] \simeq \mathbb{C}[a, b, c, d, \frac{1}{ad-bc}, \frac{1}{b}] \Big|_{f(ka, kb, kc, kd) = f(a, b, c, d)}$$

, i.e. it consists of homogeneous polynomials in  $a, b, c, d, \frac{1}{ad-bc}, \frac{1}{b}$  of degree zero.

Now

$$\begin{aligned} \mathcal{Y}' &= \{(\gamma_0, \gamma_1, z) \mid \gamma_0(0) \neq \gamma_1(0) \neq z \neq \gamma_0(0); z \in \mathbb{P}^1, \gamma_i \in H\} \\ &= \left\{ (\gamma_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \gamma_1 = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}, z = (z_0, z_1)) \mid \frac{b}{d} \neq \frac{b'}{d'} \neq \frac{z_0}{z_1} \neq \frac{b}{d} \right\} \\ &\simeq \mathbb{C}[a, b, c, d, a', b', c', d', z_0, z_1, \frac{1}{ad-bc}, \frac{1}{a'd'-b'c'}, \frac{1}{b'd-bd'}, \frac{1}{bz_1-dz_0}, \frac{1}{b'z_1-d'z_0}] \end{aligned}$$

And now

$$\begin{aligned} \underline{\mathcal{Y}}' &= H \setminus \{(\gamma_0, \gamma_1, z) \mid \gamma_0(0) \neq \gamma_1(0) \neq z \neq \gamma_0(0); z \in \mathbb{P}^1, \gamma_i \in H\} \\ &\simeq \{(1, \gamma'_1 = \gamma_0^{-1}\gamma_1, \gamma_0^{-1}(z) \mid \gamma'_1(0) \neq 0, \gamma_0^{-1}(z) \neq 0, \gamma'_1(0) (= \gamma_0^{-1}\gamma_1(z)) \neq \gamma_0^{-1}(z)\} \end{aligned}$$

As  $(\gamma'_1, \gamma_0^{-1}(z)) = \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, (z_0, z_1) \right)$  vary along  $H \times \mathbb{P}^1$ , we calculate

$$\mathbb{C}[\underline{\mathcal{Y}}'] \simeq \mathbb{C}[a, b, c, d, \frac{1}{ad-bc}, z_0, z_1, \frac{1}{b}, \frac{1}{z_0}, \frac{1}{bz_1-dz_0}] \Big|_0$$

, i.e. the homogeneous polynomials of degree zero of the variables  $a, b, c, d, \dots, \frac{1}{bz_1-dz_0}$ .

## 2 References

**Kazhdan-Lusztig I-IV** D. Kazhdan and G. Lusztig, *Tensor Structures Arising in Affine Lie Algebras*, Journal of the American Mathematical Society, 1993-1994

**Drinfeld 1989-1990** V. G. Drinfeld, papers on quasi-Hopf algebras, *Algebra i Analiz 1989-1990* (Russian), or *Leningrad Math Journal 1990-91* (English translation).