Notes on the Kazhdan-Lusztig fusion product

Yau Kwan Kiu

March 12, 2009

In the category $\mathscr{O}_\kappa$ of representations of the affine Kac-Moody algebra $\hat{\mathfrak{g}}$ with finite-dimensional weight subspaces with weights bounded from above, the fusion tensor product of two modules $V_1 \dot{\otimes} V_2$ represents the functor (???) $X \mapsto Hom_\Gamma(V_1 \otimes V_2 \otimes X, \mathbb{C})$ where X is a smooth module (see proposition 7.10, p. 941, Kazhdan-Lusztig I), and Γ is the affine Lie algebra with coefficients analytic functions on a Riemann sphere with points removed.

We work out the fusion tensor product, as described in section 4 in Kazhdan-Lusztig I, in the simple example of $V_1 \otimes V_2$, with the curve C being the Riemann sphere minus the set of three points $S = \{0, 1, \infty\}$, $R = \mathbb{C}[t, -1/t, 1/(1-t)]$, $\heartsuit = \{s_0 = \infty\}$, $\spadesuit = \{s_1 = 0, s_2 = 1\}$.

We hope that this straight-forward exercise would help make the Kazhdan-Lusztig series of papers more accessible.

1 Definitions

We follow the definitions in section 4 of [Kazhdan-Lusztig I].

Let $S = \{0, 1, \infty\}$, $\heartsuit = \{s_0 = \infty\}$, $\spadesuit = \{s_1 = 0, s_2 = 1\}$. We have $\hat{\mathfrak{g}}^\heartsuit = \hat{\mathfrak{g}}$ and $\hat{\mathfrak{g}}^\spadesuit = \mathfrak{g} \hat{\oplus} \mathfrak{g}$. Let Let $R = \mathbb{C}[t, \frac{-1}{t}, \frac{1}{1-t}]$; for $f, g \in R$, we have the residue pairing $\int f dg$. We have charts $\gamma_0 : \mathbb{C} \to C, x \mapsto (-1/x), (-1/t) \mapsto \epsilon$; $\gamma_1 : t \mapsto \epsilon$, $\gamma_2 : 1/(1-t) \mapsto \epsilon$. Let $\gamma = R \otimes \mathfrak{g} \oplus \mathbb{C} \cdot 1$ be the affine Lie algebra with R-coefficients

For the approximation property of Γ [4.9(a),p.927, Kazhdan-Lusztig I], given an n-tuple of power series $\omega = (\omega_s)_{s \in \heartsuit}$, we construct an ϵ^N-approximation of it in the space R of analytic functions on a Riemann sphere with points (a,b,c,...,p,q) removed (none of which being ∞) by

$$\sum_{q=a,b,c,\ldots,p,q} \frac{(x-a)(x-b)(x-c)\cdots(x-p)}{(q-a)(q-b)(q-c)\cdots(q-p)} \cdot (\omega_q)_N$$

where $(\omega_q)_N$ is the polynomial with the terms of ω_q up to the N-th power. If a of the points (a,b,c...,p,q) is ∞, and let's say (b,c)=(0,1) we may use the transformation $x = \frac{(t-b)(c-a)}{(t-a)(c-b)}$, or, $t = \frac{a(c-b)x-b(c-a)}{(c-b)x-(c-a)}$ which sends $(x = \infty, 0, 1) \mapsto (t = a, b, c)$.

2 A worked example

Consider the maps (4.6(a,b)): $\Gamma \to \hat{\mathfrak{g}}^{\heartsuit}$, $\Gamma \to \hat{\mathfrak{g}}^{\spadesuit}$. In our example let us take $f = \frac{1}{1-t}$ and $c \in \mathfrak{g}$. Then

$$(a) : fc = 1/(1-t)c = \frac{-1/t}{1-1/t} \mapsto \frac{\epsilon}{1+\epsilon}c = (\epsilon - \epsilon^2 + \epsilon^3 - \cdots)c \in \hat{\mathfrak{g}}^{\heartsuit}$$

$$(b) : fc = 1/(1-t)c \mapsto \big((1+t+t^2+\cdots)c, 1/(1-t)c\big) \mapsto \big((1+\epsilon+\epsilon^2+\cdots)c, \epsilon c\big) \in \hat{\mathfrak{g}}^{\spadesuit}$$

The space R_1 consists of analytic functions on our curve vanishing of order one at every point in $\heartsuit$. In our case, $\heartsuit = \{\infty\}$, so $R_1 = t^{-1}\mathbb{C}[t^{-1}]$. G_N is spanned by elements in Γ of the form $(t^{-1}c_1)(t^{-1}c_2)\cdots(t^{-1}c_N)$.

The limit module $\hat{W} = \varprojlim_N W/G_N W$ can be thought of as the space of infinite series $\{x = w_0 + p_1 w_1 + p_2 w_2 + \cdots\}$ where $p_n \in G_n$ and $w \in W$.

The space T(W) is defined as $\hat{W}(-\infty)$, i.e. those vector-series which are killed by some $Q_N^{\#} = span\{(\epsilon^{-1}c_1)\cdots(\epsilon^{-1}c_N)\}$ and the fusion product $T(W)^{\#}$ is defined via the involution $\# : \epsilon \mapsto (-1/\epsilon), \mathbf{1} \mapsto -\mathbf{1}$. Going the other way, we may define $T(W)^{\#} = (\hat{W}^{\#})(\infty)$. In our case, via our chart γ_0, we have $\epsilon c \mapsto (-1/t)c$, which is analytically continued to expressions $(\frac{-1}{t}c, (\frac{-1}{1-1/\frac{1}{1-t}})c) = (\frac{-1}{\epsilon}c, (\frac{-1}{1-\epsilon})c) = \big(\frac{-1}{\epsilon}c, -(1+\epsilon+\epsilon^2+\cdots)c\big)$ at the points $s_1 = 0$ and $s_2 = 1$. Then $\epsilon^{-1}c \mapsto (-t)c$, which is analytically continued to $(-tc, ((1-t)-1)c) = (-tc, (\frac{1}{1/(1-t)} - 1)c) \mapsto \big(-\epsilon c, (\frac{1}{\epsilon} - 1)c\big)$ at the points $s_1 = 0$ and $s_2 = 1$.

To construct an example of an element in the fusion tensor product, we may take $v_1 \otimes v_2 \in V_1 \otimes V_2$, where v_1 is a highest weight vector in V_1 and v_2 is of the form $(1 + \epsilon^{-1} + \epsilon^{-2} + \cdots)c'u$, where $u \in V_2$ and $c' \in \mathfrak{g}$. Then $v_1 \otimes v_2$

is killed by the image of $\epsilon^{-1}c \in \hat{\mathfrak{g}}^{\heartsuit}$ in $\hat{\mathfrak{g}}^{\spadesuit}$, for the points $s_1 = 0, s_2 = 1$ in the map just above.

And for smooth modules V_1, V_2, X and $W = V_1 \otimes V_2$, there is a map (4.18(a)) of coinvariants: $(T(W) \otimes X)_{\tilde{\mathfrak{g}}^{\heartsuit}} \to (W \otimes X)_{\Gamma}$ which in our notation sends

$$(w_0 + p_1 w_1 + p_2 w_2 + \cdots) \otimes x \mapsto w_0 \otimes x + w_1 p_1 \otimes x + \cdots$$

The right-hand side of the map, defined modulo $\Gamma(W \otimes X)$, is a finite sum because of 4.15(a), and X is smooth.

3 Finiteness of coinvariants

Recall that, in our example, $C = \mathbb{P}^1 - \{\infty, 0, 1\}, S = \{s_0 = \infty, s_1 = 0, s_2 = 1\}, \heartsuit = \{s_0 = \infty\}; [s_0] = \{0, 1\}; \gamma_{s_0} = (t \mapsto -(1/t)), \gamma_{s_1} = (t \mapsto t), \gamma_{s_2} = (t \mapsto 1/(1-t)); R = \mathbb{C}[t, 1/t, 1/(1-t)]$.

Following [section 7.2, Kazhdan-Lusztig I], we define functions on the complex curve C: For $s = s_1 = 0, p \in C$, we have $g_s = g_0 = g_0(p) = \frac{1}{1/\gamma_{s_0}^{-1}(p) - 1/\gamma_{s_0}^{-1}(\gamma_s(0))} = \frac{1}{1/\frac{-1}{p} - 1/\frac{-1}{0}} = -1/p$. For $s = s_2 = 1, p \in C$, we have $g_s = g_1(p) = \frac{1}{1/(\frac{-1}{p}) - 1/(\frac{-1}{1})} = \frac{1}{-p+1} = \frac{1}{1-p}$.

Now when $s = s_1 = 0$, we have

$$g_0(t) = -1/t.$$

Using our charts γ_{s_i}, its expansions at $s_i = \infty, 0, 1$ are (notations as in §7.2(a)):

$$^{\infty}g_0 = -1/t = \epsilon \in \epsilon + \epsilon^2\mathbb{C}[[\epsilon]]$$

$$^{0}g_0 = -1/t = -1/\epsilon = r\epsilon^{-1} + r'$$

$$^{1}g_0 = -1/t = \frac{1}{1 - \frac{1}{1-t}} - 1 = \frac{1}{1-\epsilon} - 1 = \epsilon + \epsilon^2 + \epsilon^3 + \cdots$$

When $s = s_2 = 1$, we have

$$g_1(t) = 1/(1-t).$$

Its expansions at $\infty, 0, 1$ are:

$$^{\infty}g_1 =^{\infty} (1/(1-t)) =^{\infty} \left(\frac{1/t}{1/t - 1}\right) = -(1/t) - (1/t)^2 - (1/t)^3 - \cdots = \epsilon - \epsilon^2 + \epsilon^3 - \cdots \in \epsilon + \epsilon^2\mathbb{C}[[\epsilon]]$$

$$ {}^0g_1 = {}^0 (1/(1-t)) = 1 + t + t^2 + \cdots = 1 + \epsilon + \epsilon^2 + \cdots \in \mathbb{C}[[\epsilon]] $$
$$ {}^1g_1 = {}^1 (1/(1-t)) = 1/1 - t = \epsilon $$

(¡– ??? but : c.f. 7.2(a) - doesn't fit the equation???)

Now suppose that the modules V_0, V_1 are highest weight modules of highest weight λ_0, λ_1 of level k. Specifically, we take the Weyl modules $V_{\lambda_i} = U(\tilde{\mathfrak{g}}_-) \otimes_{U(\mathfrak{g})} L_{\lambda_i}$, where L_{λ_i} is an irreducible module of $\mathfrak{g}$ with highest weight λ. Then we can take $N_0, N_1 = 1$ and $W_1 = V_0(1) \otimes V_1(1) = L_{\lambda_0} \otimes L_{\lambda_1} \subset W = V_0 \otimes V_1$.

As before, $R = \mathbb{C}[t, t^{-1}, \frac{1}{1-t}]$, $R_1 = t^{-1}\mathbb{C}[t^{-1}]$, and $G_1 = R_1 \otimes \mathfrak{g}$. We have $G_1 W = t^{-1}\mathbb{C}[t^{-1}]\mathfrak{g}W$.

Let us take as before $f = g_0 = -1/t$ (section 7.2) and $fc = -t^{-1}c \in G_1$, where $c \in \mathfrak{g}$ and consider its action on $v_0 \otimes v_1 \in L_{\lambda_0} \otimes L_{\lambda_1}$.

Via the charts γ_s, we have the expansions of f, respectively at $s = \infty, 0, 1$:

$$ {}^\infty f = -1/t = \epsilon $$
$$ {}^0 f = -1/t = -\epsilon^{-1} $$
$$ {}^1 f = -1/t = \frac{1}{1 - \frac{1}{1-t}} - 1 = \frac{1}{1-\epsilon} - 1 = \epsilon + \epsilon^2 + \epsilon^3 + \cdots $$

So

$$ \begin{aligned} (-t^{-1}c)(v_0 \otimes v_1) &= (-\epsilon^{-1}c)(v_0) \otimes v_1 + v_0 \otimes ((\epsilon + \epsilon^2 + \epsilon^3 + \cdots)c)(v_1) \\ &= (-\epsilon^{-1}c)(v_0) \otimes v_1 \end{aligned} $$

And we see that $(-\epsilon^{-1}c)(v_0) \otimes v_1 \in G_1W$. Continuing in this manner, we see that $(V_0 - V_0(1)) \otimes (V_1(1)) \subset G_1 W$.

For a general $v_0 \otimes v_1 \in V_0 \otimes V_1$, we can rearrange the first equality above to get

$$ (-\epsilon^{-1}c)(v_0) \otimes v_1 = -(-t^{-1}c)(v_0 \otimes v_1) + v_0 \otimes ((\epsilon + \epsilon^2 + \epsilon^3 + \cdots)c)(v_1) $$

where the right-hand side is a finite sum (V_1 is a highest weight module and thus it is smooth). The first term on the right-hand side is in G_1W and the second term on the right-hand side is of degrees higher than that of the left-hand side. We can solve recursively for $(-\epsilon^{-1}c)(v_0) \otimes v_1 + v_0 \otimes (\epsilon c)(v_1)$ on the right-hand side and so we can recursively express $v_0 \otimes v_1$ as a sum of elements of "higher" ϵ-degrees and elements from G_1W. We thus conclude that $W = V_0(1) \otimes V_1(1) + G_1 W$, as is stated in more general terms in Proposition 7.4, pp.938-940.

4 References

Kazhdan-Lusztig I-IV D. Kazhdan and G. Lusztig, *Tensor Structures Arising in Affine Lie Algebras*, Journal of the American Mathematical Society, 1993-1994

Drinfeld 1989-1990 V. G. Drinfeld , papers on quasi-Hopf algebras, *Algebra i Analiz 1989-1990* (Russian), or *Leningrad Math Journal 1990-91* (English translation).