Exercises on coinvariant bundles (work in progress)

Yau Kwan Kiu

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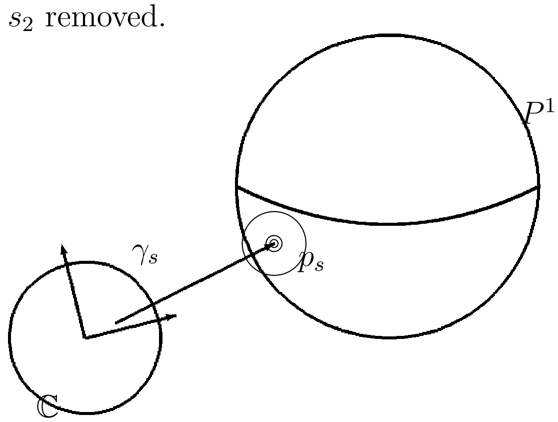
We continue the fusion exercise in `fusionex.tex` and study the Kazhdan-Lusztig fusion product $\dot{\otimes}$ in the simple case over the Riemann sphere with three points s_0, s_1, s_2 removed.



1 The parameter space

When $s_0 = \infty, s_1 = 0, s_2 = 1$, we can study the modules over the Lie algebra $\mathfrak{g}(\mathbb{C}[1/t, 1/(1-t)])$ with coefficients regular functions over $\mathbb{P}^1 - \{0, 1\}$.

We now allow the points s_0, s_1, s_2 to move. Consider a map $s : \mathbb{P}^1 \to \mathbb{P}^1 : s(s_1) = 0, s(s_2) = 1$. We then have the Lie algebra $\mathfrak{g}(\mathbb{C}[\mathbb{P}^1 - \{s_1, s_2\}]) =$

$\mathfrak{g}(\mathbb{C}[1/s, 1/(1-s)])$.

Following Kazhdan-Lusztig II, §7, we let $H = PGL_2(\mathbb{C})$ be the group of morphisms of the projective line $\mathbb{P}^1(\mathbb{C})$.

We continue our simple scenario in [reference: fusionex], and let $S = \{s_0, s_1, s_2\}, \heartsuit = \{s_0\}, \spadesuit = \{s_1, s_2\}, C = \mathbb{P}^1 - \{s_0, s_1, s_2\}$; Let V_1, V_2 be highest weight modules (e.g. Weyl modules $V_{\lambda_1}, V_{\lambda_2}$ over $\tilde{\mathfrak{g}}$ generated by irreducible $\mathfrak{g}$-modules L_{λ_i}).

Over on $\mathbb{P}^1$ we consider the parameter space $\mathscr{V} = \{(\gamma_1, \gamma_2) | \gamma_1(0) \neq \gamma_2(0)\}$ of two charts $\gamma_1, \gamma_2 : (\mathbb{P}^1 \to \mathbb{P}^1) \in H$ at distinct positions on the projective line, and the parameter space $\mathscr{V}' = \{(\gamma_1, \gamma_2, z) | \gamma_1(0), \gamma_2(0), z \textit{ all distinct} \in \mathbb{P}^1\}$ of two charts plus a point $z \in \mathbb{P}^1$ at distinct positions on the projective line.

1.1 Spaces of functions over the parameter spaces

Now let us calculate the spaces of functions $\mathbb{C}[\mathscr{V}], \mathbb{C}[\mathscr{V}'], \mathbb{C}[\underline{\mathscr{V}}], \mathbb{C}[\underline{\mathscr{V}'}]$.

First of all, we have

$$\begin{aligned} \mathbb{C}[H] &= \mathbb{C}[PGL_2(\mathbb{C})] \\ &= \mathbb{C}[a, b, c, d, \frac{1}{ad-bc}]|_{f(ka,kb,kc,kd)=f(a,b,c,d)} \end{aligned} \tag{1}$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H$ i.e. a regular function on H is a homogeneous polynomial in $a, b, c, d, 1/(ad-bc)$ with degree zero. We shall denote the space of such functions also by $\mathbb{C}[a, b, c, d, \frac{1}{ad-bc}]|_{(0)}$.

Next we have

$$\mathbb{C}[H \times H] = \mathbb{C}[a, b, c, d, a', b', c', d', \frac{1}{ad-bc}, \frac{1}{a'd'-b'c'}]|_{f(ka,kb,kc,kd,ma',mb',mc',md')=f(a,b,c,d,a',b',c',d')} \tag{2}$$

Next we have $\mathscr{V} \subset H$. The condition for a pair of charts γ_1, γ_2 to be in $\mathscr{V}$ is that their images at zero are distinct, i.e. $b/d \neq b'/d'$, or $b'd - d'b \neq 0$, where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \gamma_1$, $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \gamma_2$. So we have

$$\tilde{A} = \mathbb{C}[\mathscr{V}] = \mathbb{C}[a, b, c, d, a', b', c', d', \frac{1}{ad-bc}, \frac{1}{a'd'-b'c'}, \frac{1}{b'd-d'b}]|_{(0,0)} \tag{3}$$

,i.e. polynomials in $a, b, ..., \frac{1}{b'd-d'b}$, homogeneous of degree zero in both (a, b, c, d) and (a', b', c', d').

To calculate $A = \mathbb{C}[\underline{\mathscr{V}}]$, we may take the subring of H-invariants of $\tilde{A}$ calculated above. Alternatively, consider

$$\begin{aligned} \underline{\mathscr{V}} = H\backslash\{(\gamma_0, \gamma_1)|\gamma_0(0) \neq \gamma_1(0)\} &\simeq \{(1, \gamma_1' = \gamma_0^{-1}\gamma_1)|\gamma_1'(0) \neq 0 \in H\} \\ &\simeq \left\{\gamma_1' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H | \frac{b}{d} \neq \frac{0}{1}\right\} \end{aligned}$$

Then $\underline{\mathscr{V}} \subset H$ and so

$$A = \mathbb{C}[\underline{\mathscr{V}}] \simeq \mathbb{C}[a, b, c, d, \frac{1}{ad-bc}, \frac{1}{b}]|_{f(ka,kb,kc,kd)=f(a,b,c,d)}, \tag{4}$$

i.e. it consists of homogeneous polynomials in $a, b, c, d, \frac{1}{ad-bc}, \frac{1}{b}$ of degree zero.

Now

$$\begin{aligned} \mathscr{V}' &= \left\{(\gamma_0, \gamma_1, z)|\gamma_0(0) \neq \gamma_1(0) \neq z \neq \gamma_0(0); z \in \mathbb{P}^1, \gamma_i \in H\right\} \\ &= \left\{(\gamma_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \gamma_1 = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}, z = (z_0, z_1))|\frac{b}{d} \neq \frac{b'}{d'} \neq \frac{z_0}{z_1} \neq \frac{b}{d}\right\} \end{aligned}$$

So

$$\tilde{A}' \simeq \mathbb{C}[a, b, c, d, a', b', c', d', z_0, z_1, \frac{1}{ad-bc}, \frac{1}{a'd'-b'c'}, \frac{1}{b'd-bd'}, \frac{1}{bz_1-dz_0}, \frac{1}{b'z_1-d'z_0}] \tag{5}$$

And now

$$\begin{aligned} \underline{\mathscr{V}}' &= H\backslash\left\{(\gamma_0, \gamma_1, z)|\gamma_0(0) \neq \gamma_1(0) \neq z \neq \gamma_0(0); z \in \mathbb{P}^1, \gamma_i \in H\right\} \\ &\simeq \left\{(1, \gamma_1' = \gamma_0^{-1}\gamma_1, \gamma_0^{-1}(z)|\gamma_1'(0) \neq 0, \gamma_0^{-1}(z) \neq 0, \gamma_1'(0) \left(= \gamma_0^{-1}\gamma_1(z)\right) \neq \gamma_0^{-1}(z)\right\}. \end{aligned}$$

As $(\gamma_1', \gamma_0^{-1}(z)) = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (z_0, z_1)\right)$ vary along $H \times \mathbb{P}^1$, we calculate

$$A' = \mathbb{C}[\underline{\mathscr{V}}'] \simeq \mathbb{C}[a, b, c, d, \frac{1}{ad-bc}, z_0, z_1, \frac{1}{b}, \frac{1}{z_0}, \frac{1}{bz_1-dz_0}]|_{degree=0} \tag{6}$$

i.e. the polynomials in the variables $a, b, c, d, ..., \frac{1}{bz_1-dz_0}$, homogeneous of degree zero in (a, b, c, d) and (z_0, z_1).

Following section 9.3, let us take the function

$$f = \frac{az_1 - cz_0}{bz_1 - dz_0} \in \mathbb{C}[\underline{\mathscr{V}}] \tag{7}$$

where, as we recall from equation (4), a, b, c, d are matrix components of $\gamma_1 \in H = PGL_2(\mathbb{C})$ and (z_0, z_1) are the homogeneous coordinates of a point in $\mathbb{P}^1$, and we have the embedding $\mathbb{C} \to \mathbb{P}^1 : z_0/z_1 \mapsto (z_0, z_1)$. Now for $s = 0, 1$,

$$^s f = \sum_n p_{n,s} \epsilon^n = f(\gamma_s(z))$$

For $s = 0$, we have chosen $\tilde{u} = (\gamma_0, \gamma_1)$ such that $\gamma_0 = \mathbf{1}$ [c.f. equation (4)]and so f can be expanded at $z_0/z_1 = z = 0$ as:

$$\begin{aligned} {}^0 f = \frac{az_1 - cz_0}{bz_1 - dz_0} = \frac{a - cz}{b - dz} &= \frac{a - cz}{b} \frac{1}{1 - (dz/b)} \\ &= \frac{a - cz}{b}[1 + (dz/b) + (dz/b)^2 + (dz/b)^3 + \cdots] \end{aligned} \tag{8}$$

For $s = 1$, $\gamma_1(z) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \left(\frac{z_0}{z_1} \right) = \frac{az_0 + bz_1}{cz_0 + dz_1}$, and so from (7),

$$^1 f = \frac{a(cz_0 + dz_1) - c(az_0 + bz_1)}{b(cz_0 + dz_1) - d(az_0 + bz_1)} = \frac{(ad - bc)z_1}{(bc - ad)z_0} = \frac{-z_1}{z_0} \tag{9}$$

!!!???CHECK THE ABOVE???!!!

Now if we follow the recipe of 9.5 to define a basis of A' over A, we get

$$f_0 = \frac{1}{\gamma_0^{-1}(z)} = \frac{z_1}{z_0} \tag{10}$$

since, as we recall from (6), (z_0, z_1) are the homogeneous coordinates for $\gamma_0^{-1}(z)$ and we identify $\gamma_0^{-1}(z) = z_0/z_1$ over $\mathbb{C}$; and

$$f_1 = \frac{1}{\gamma_1^{-1}(z)} = -(ad - bc) \frac{az_1 - cz_0}{bz_1 - dz_0} \tag{11}$$

which happens to be a scalar multiple of the f that we have just studied above in (9).

When we specialise at $(a, b, c, d) = (0, 1, -1, 1)$, we get the simple assertion that the positive powers of

$$\begin{aligned} f_0 &= z_1/z_0 \\ f_1 &= -\frac{z_0}{z_1 - z_0} = \frac{z_0/z_1}{1 - z_0/z_1} \end{aligned} \tag{12}$$

form a basis of

$$A'|_{(a,b,c,d)=(0,1,-1,1)} = \mathbb{C}[z_0, z_1, \frac{1}{z_0}, \frac{1}{z_1 - z_0}]|_{(0)}$$

(c.f. (6)) over

$$A = \mathbb{C}$$

(c.f. (4)).

2 References

Kazhdan-Lusztig I-IV D. Kazhdan and G. Lusztig, *Tensor Structures Arising in Affine Lie Algebras*, Journal of the American Mathematical Society, 1993-1994

Drinfeld 1989-1990 V. G. Drinfeld , papers on quasi-Hopf algebras, *Algebra i Analiz 1989-1990* (Russian), or *Leningrad Math Journal 1990-91* (English translation).

kl-index *Index of notations used by Kazhdan-Lusztig*, in this series of notes

fusionex *Notes on the Kazhdan-Lusztig fusion product*, in this series of notes

coinbundle *Exercises on the bundle of coinvariants*, in this series of notes