

Exercises on the Kazhdan-Lusztig fusion product

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In the category \mathcal{O}_κ of representations of the affine Kac-Moody algebra $\hat{\mathfrak{g}}$ with finite-dimensional weight subspaces with weights bounded from above, the fusion tensor product of two modules $V_1 \hat{\otimes} V_2$ represents the functor (???) $X \mapsto \text{Hom}_\Gamma(V_1 \otimes V_2 \otimes X, \mathbb{C})$ where X is a smooth module (see proposition 7.10, p. 941, Kazhdan-Lusztig I), and Γ is the affine Lie algebra with coefficients analytic functions on a Riemann sphere with points removed.

We work out the fusion tensor product, as described in section 4 in Kazhdan-Lusztig I, in the simple example of $V_1 \otimes V_2$, with the curve C being the Riemann sphere minus the set of three points $S = \{0, 1, \infty\}$, $R = \mathbb{C}[t, -1/t, 1/(1-t)]$, $\heartsuit = \{s_0 = \infty\}$, $\spadesuit = \{s_1 = 0, s_2 = 1\}$.

1 Definitions

We follow the definitions in section 4 of [Kazhdan-Lusztig I].

Let $S = \{0, 1, \infty\}$, $\heartsuit = \{s_0 = \infty\}$, $\spadesuit = \{s_1 = 0, s_2 = 1\}$. We have $\hat{\mathfrak{g}}^\heartsuit = \hat{\mathfrak{g}}$ and $\hat{\mathfrak{g}}^\spadesuit = \mathfrak{g} \hat{\oplus} \mathfrak{g}$. Let $R = \mathbb{C}[t, \frac{-1}{t}, \frac{1}{1-t}]$; for $f, g \in R$, we have the residue pairing $\int f dg$. We have charts $\gamma_0 : \mathbb{C} \rightarrow C, x \mapsto (-1/x), (-1/t) \mapsto \epsilon$; $\gamma_1 : t \mapsto \epsilon, \gamma_2 : 1/(1-t) \mapsto \epsilon$. Let $\gamma = R \otimes \mathfrak{g} \oplus \mathbb{C} \cdot 1$ be the affine Lie algebra with R -coefficients

For the approximation property of Γ [4.9(a), p.927, Kazhdan-Lusztig I], given an n -tuple of power series $\omega = (\omega_s)_{s \in \heartsuit}$, we construct an ϵ^N -approximation of it in the space R of analytic functions on a Riemann sphere with points

(a,b,c,\dots,p,q) removed (none of which being ∞) by

$$\sum_{q=a,b,c,\dots,p,q} \frac{(x-a)(x-b)(x-c)\cdots(x-p)}{(q-a)(q-b)(q-c)\cdots(q-p)} \cdot (\omega_q)_N$$

where $(\omega_q)_N$ is the polynomial with the terms of ω_q up to the N-th power. If a of the points (a,b,c,\dots,p,q) is ∞ , and let's say $(b,c)=(0,1)$ we may use the transformation $x = \frac{(t-b)(c-a)}{(t-a)(c-b)}$, or, $t = \frac{a(c-b)x-b(c-a)}{(c-b)x-(c-a)}$ which sends $(x = \infty, 0, 1) \mapsto (t = a, b, c)$.

2 A worked example

Consider the maps (4.6(a,b)): $\Gamma \rightarrow \hat{\mathfrak{g}}^\heartsuit$, $\Gamma \rightarrow \hat{\mathfrak{g}}^\spadesuit$. In our example let us take $f = \frac{1}{1-t}$ and $c \in \mathfrak{g}$. Then

$$(a) : fc = 1/(1-t)c = \frac{-1/t}{1-1/t} \mapsto \frac{\epsilon}{1+\epsilon}c = (\epsilon - \epsilon^2 + \epsilon^3 - \dots)c \in \hat{\mathfrak{g}}^\heartsuit$$

$$(b) : fc = 1/(1-t)c \mapsto ((1+t+t^2+\dots)c, 1/(1-t)c) \mapsto ((1+\epsilon+\epsilon^2+\dots)c, \epsilon c) \in \hat{\mathfrak{g}}^\spadesuit$$

The space R_1 consists of analytic functions on our curve vanishing of order one at every point in \heartsuit . In our case, $\heartsuit = \{\infty\}$, so $R_1 = t^{-1}\mathbb{C}[t^{-1}]$. G_N is spanned by elements in Γ of the form $(t^{-1}c_1)(t^{-1}c_2)\cdots(t^{-1}c_N)$.

The limit module $\hat{W} = \varprojlim_N W/G_N W$ can be thought of as the space of infinite series $\{x = w_0 + p_1 w_1 + p_2 w_2 + \dots\}$ where $p_n \in G_n$ and $w \in W$.

The space $\mathbb{T}(W)$ is defined as $\hat{W}(-\infty)$, i.e. those vector-series which are killed by some $Q_N^\# = \text{span}\{(\epsilon^{-1}c_1)\cdots(\epsilon^{-1}c_N)\}$ and the fusion product $T(W)^\#$ is defined via the involution $\# : \epsilon \mapsto (-1/\epsilon)$, $\mathbf{1} \mapsto -\mathbf{1}$. Going the other way, we may define $T(W)^\# = (\hat{W}^\#)(\infty)$. In our case, via our chart γ_0 , we have $\epsilon c \mapsto (-1/t)c$, which is analytically continued to expressions $(\frac{-1}{t}c, (\frac{-1}{1-1/\epsilon})c) = (\frac{-1}{\epsilon}c, (\frac{-1}{1-\epsilon})c) = (\frac{-1}{\epsilon}c, -(1+\epsilon+\epsilon^2+\dots)c)$ at the points $s_1 = 0$ and $s_2 = 1$. Then $\epsilon^{-1}c \mapsto (-t)c$, which is analytically continued to $(-tc, ((1-t)-1)c) = (-tc, (\frac{1}{1/(1-t)}-1)c) \mapsto (-\epsilon c, (\frac{1}{\epsilon}-1)c)$ at the points $s_1 = 0$ and $s_2 = 1$.

To construct an example of an element in the fusion tensor product, we may take $v_1 \otimes v_2 \in V_1 \otimes V_2$, where v_1 is a highest weight vector in V_1 and v_2 is of the form $(1+\epsilon^{-1}+\epsilon^{-2}+\dots)c'u$, where $u \in V_2$ and $c' \in \mathfrak{g}$. Then $v_1 \otimes v_2$ is killed by the image of $\epsilon^{-1}c \in \hat{\mathfrak{g}}^\heartsuit$ in $\hat{\mathfrak{g}}^\spadesuit$, for the points $s_1 = 0, s_2 = 1$ in the map just above.

3 The duality between $T(W)$ and $T'(W)$

4 References

Kazhdan-Lusztig I-IV D. Kazhdan and G. Lusztig, *Tensor Structures Arising in Affine Lie Algebras*, Journal of the American Mathematical Society, 1993-1994

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