1. Let $f(x, y) = x^2 + 4x + y^2 + 4y$ and $R = \{x^2 + y^2 \leq 18\}$.

   (a) Does f necessarily have absolute maxima and/or minima on R? Explain.

   (b) Find all critical points of f in the interior of the region R.

   (c) Classify each critical point as relative maximum, relative minimum, or saddle point where applicable.

   (d) Find the absolute maximum and/or the minimum value of f on R.

   (e) Let $R' = \{x^2 + y^2 \leq 17.9997\}$. Estimate the absolute maximum and/or the minimum value of f on R'.

   (a) f is a continuous function (since it is a polynomial), and R is a closed and bounded region. By the Extreme Value Theorem, f attains both absolute maximum and absolute minimum on R.

   (b) $\nabla f = 0$ or ∇f undefined. Since f is a polynomial, its gradient is always defined.

$$\begin{aligned}\nabla f &= 0\\ (2x+4)\hat{i} + (2y+4)\hat{j} &= 0\\ (2x+4) &= 0\\ (2y+4) &= 0\end{aligned}$$

   So $x = -2$ and $y = -2$. Need to check that the point is in the interior of R:

$$\begin{aligned}x^2 + y^2 &< 18\\ (2)^2 + (2)^2 &< 18\\ 8 &< 18\end{aligned}$$

   Since 8 is less than 18, it follows that $(-2, -2)$ is in the interior of R.
   $(-2, -2)$ (1pt - writing the actual point as the final answer.)

   (c) There are at least two ways to do this part, and all of them are equally acceptable.
   Method 1. Second derivative test.

$$\begin{aligned}D &= f_{xx}f_{yy} - f_{xy}^2\\ &= (2x+4)_x(2y+4)_y - (2x+4)_y^2\\ &= 2 * 2 - 0 \text{ (1pt) - Correct partial derivatives}\\ D &= 4 > 0\end{aligned}$$

   $D > 0$ and $f_{xx} > 0$, so $(-2, -2)$ is a relative minimum.
   Method 2. Completing the Square.

$$\begin{aligned}f(x, y) &= x^2 + 4x + y^2 + 4y\\ &= (x^4 + 4x + 4) + (y^2 + 4y + 4) - 8\\ &= (x+2)^2 + (y+2)^2 - 8\end{aligned}$$

(d) Must use Lagrange Multiplier:

$$\begin{aligned}
\nabla f &= \lambda \nabla g \\
2x + 4 &= 2x\lambda \\
2y + 4 &= 2y\lambda
\end{aligned}$$

(1pt)
$x \neq 0$ and $y \neq 0$ since otherwise we get $4 = 0$. So we can divide by x and y:

$$\begin{aligned}
\frac{x+2}{x} &= \frac{y+2}{y} \\
xy + 2y &= xy + 2x \\
y &= x
\end{aligned}$$

Since we are on the boundary of the region, we have $x^2 + y^2 = 18$.

$$\begin{aligned}
x^2 + x^2 &= 18 \\
x &= \pm 3
\end{aligned}$$

This gives us points $(3, 3)$ and $(-3, -3)$.

$$\begin{aligned}
f(-2, -2) &= -8 \\
f(3, 3) &= 42 \\
f(-3, -3) &= -6
\end{aligned}$$

So the maximum value is 42, and the minimum value is -8.

(e) The minimum is in the interior of the region, so it doesn't change. The maximum is on the boundary, so it will change.

$$\begin{aligned}
(2(3) + 4) &= 6\lambda \\
5/3 = \lambda &\approx \frac{f_{\text{new max}} - 42}{17.9997 - 18} \\
f_{\text{new max}} &\approx 41.9995
\end{aligned}$$

So the new maximum is approximately 41.9995 and the new minimum is -8.

2. Let $\vec{F}(x,y,z) = x\hat{\imath} - \frac{z}{y^2+z^2}\hat{\jmath} + \frac{y}{y^2+z^2}\hat{k}$, and let $C = \{y^2+z^2 = a^2, x = 1\}$ oriented by the right hand rule with respect to $\hat{\imath}$.

   (a) Find the curl of $\vec{F}$.

   (b) Is $\vec{F}$ conservative? Explain

   (c) Find the work done by $\vec{F}$ on a particle that starts from the point $(1, a, 0)$ and goes around C exactly once along the orientation.

(a)

$$
\begin{aligned}
\nabla \times \vec{F} &= \det \begin{pmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ x & \frac{-z}{y^2+z^2} & \frac{y}{y^2+z^2} \end{pmatrix} \\
&= \left(\frac{-2y^2}{(y^2+z^2)^2} + \frac{-2z^2}{(y^2+z^2)^2} + \frac{2}{y^2+z^2} \right) \hat{\imath} - 0\hat{\jmath} + 0\hat{k} \\
&= \vec{0}
\end{aligned}
$$

(b) $\vec{F}$ is not conservative since the line integral around the closed curve given in part c is not equal to 0.

Using the curl test. The domain of $\vec{F}$ is $\mathbb{R}^3 \setminus \{x - axis\}$, and this space is not simply connected. (That is, not every loop can be contracted to a point. In particular, any loop that goes around the x-axis.) Therefore, $\vec{F}$ fails the curl test. So no conclusion can be drawn.

(c)

$$
\begin{aligned}
\vec{r}(t) &= 1\hat{\imath} + a\cos(t)\hat{\jmath} + a\sin(t)\hat{k} \\
\int_C \vec{F} \cdot d\vec{r} &= \int_0^{2\pi} (x\hat{\imath} + \frac{-z}{y^2+z^2}\hat{\jmath} + \frac{y}{y^2+z^2}\hat{k}) \cdot (-a\sin(t)\hat{\jmath} + a\cos(t)\hat{k}) dt \\
&= \int_0^{2\pi} (1\hat{\imath} + \frac{-a\sin(t)}{a^2}\hat{\jmath} + \frac{a\cos(t)}{a^2}\hat{k}) \cdot (-a\sin(t)\hat{\jmath} + a\cos(t)\hat{k}) dt \\
&= \int_0^{2\pi} 1 dt = 2\pi
\end{aligned}
$$

3. Let $\vec{F} = \hat{\imath} + \hat{k}$. Find the flux of $\vec{F}$ through the surface described by $f(x, y) = \sqrt{x^2 + y^2}$ with $0 \leq z \leq b$.

$$-\int_S \vec{F} \cdot d\vec{A} = \int_{\text{cone}} \vec{F} \cdot d\vec{A} = \int_{\text{cone with top}} \vec{F} \cdot d\vec{A} - \int_{\text{disk}} \vec{F} \cdot d\vec{A}$$

Make note of orientation. The surface S is a function, so its orientation is "upward." Cone should have "downward", the cone with bottom should have "outward", and so the disk should have "up" (unless it's a plus instead of minus.)

$$\int_{\text{cone with bottom}} (\hat{\imath} + \hat{k}) \cdot d\vec{A} = \int_{\text{solid cone}} 0 dV = 0$$
$$\int_{\text{disk}} (\hat{\imath} + \hat{k}) \cdot (\hat{k}) dA = \int_{\text{disk}} dA = \pi b^2$$

$$Flux = -(0 - \pi b) = \pi b^2$$

Method 2.

Surface integral via parametrization. There are many acceptable parametrizations, but this one certainly works:

$$\vec{r}(x, y) = x\hat{\imath} + y\hat{\jmath} + \sqrt{x^2 + y^2}\hat{k}$$

$$\begin{aligned}
\int_S \vec{F} \cdot d\vec{A} &= \int\int (\hat{\imath} + \hat{k}) \cdot (-\frac{x}{\sqrt{x^2 + y^2}}\hat{\imath} - \frac{y}{\sqrt{x^2 + y^2}}\hat{\jmath} + \hat{k}) dxdy \\
&= \int\int \left(-\frac{x}{\sqrt{x^2 + y^2}} + 1 \right) dxdy \\
&= \int_0^{2\pi} \int_0^b (-\cos\theta + 1) r dr d\theta \\
&= 2\pi \frac{b^2}{2} = \pi b^2
\end{aligned}$$

4. Let $\vec{F}(x,y,z) = y\hat{i} + x\hat{j} + zy\hat{k}$, let S be the surface of a cube of length c with one corner at the origin and three of edges on the positive x,y, and the z-axis *WITH AN OPEN TOP* .

   (a) Find $\nabla \times \vec{F}$.

   (b) Find the flux of $\nabla \times \vec{F}$ through the surface S.

   Note that unlike other problems, the points are not equally distributed.

   (a)

$$\begin{aligned}\nabla \times \vec{F} &= \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ y & x & zy \end{pmatrix} \\ &= z\hat{i}\end{aligned}$$

   (b) *Method 1*

$$\begin{aligned}\int_S z\hat{i} \cdot d\vec{A} &= \int_{\text{box with open top}} z\hat{i} \cdot d\vec{A} \\ &= \int_{\text{closed box}} z\hat{i} \cdot d\vec{A} - \int_{\text{top}} z\hat{i} \cdot d\vec{A}\end{aligned}$$

   Orientation on the boxes are outwards. Orientation on the "top" should be upwards. (Down if it's addition instead of subtraction.)

$$\int_{\text{closed bx}} z\hat{i} \cdot d\vec{A} = \int_{\text{solid box}} 0 dV = 0$$

$$\int_{\text{top}} z\hat{i} \cdot \hat{k} dA = 0$$

$$\int_{\text{box with open top}} z\hat{i} \cdot d\vec{A} = 0 - 0 = 0$$

   An equivalent method would be using Stoke's Theorem Twice (replacing the surface with the surface that shares the same boundary.) If using this method, make sure the orientation is correct.

   Method 2 Direct Computation.
   Only components of the surface that contribute to the line integral are the faces normal to the vector $\hat{i}$. So

$$\begin{aligned}\int_S z\hat{i} \cdot d\vec{A} &= \int_{x=c,0<y<c,0<z<c} z\hat{i} \cdot \hat{i} dA + \int_{x=0,0<y<c,0<z<c} z\hat{i} \cdot \hat{-i} dA \\ &= \int_{0<y<c,0<z<c} z\ dA - \int_{0<y<c,0<z<c} z\ dA = 0\end{aligned}$$

   Method 3 Stoke's Theorem (Computing the line Integral)

$$\int_S \nabla \times \vec{F} \cdot d\vec{A} = \int_C \vec{F} \cdot d\vec{A}$$

Orientation on C should be so that it is "clockwise" when looking at it from the standard xyz-plane.

$$\int_C \vec{F}\cdot d\vec{A} = \int_{C_1} \vec{F}\cdot d\vec{A} + \int_{C_2} \vec{F}\cdot d\vec{A} + \int_{C_3} \vec{F}\cdot d\vec{A} + \int_{C_4} \vec{F}\cdot d\vec{A}$$

$$\int_{C_1} \vec{F}\cdot d\vec{A} = \int_c^0 c dy$$

$$\int_{C_2} \vec{F}\cdot d\vec{A} = \int_c^0 0 dx$$

$$\int_{C_3} \vec{F}\cdot d\vec{A} = \int_0^c 0 dy$$

$$\int_{C_4} \vec{F}\cdot d\vec{A} = \int_0^c c dx$$

$$\int_C \vec{F}\cdot d\vec{A} = -c^2 + 0 + 0 + c^2 = 0$$

5. Let $\vec{F}(x,y,z) = x^2yz\hat{i} + xy^2z\hat{j} + xyz^2\hat{k}$.

(a) Find the divergence of $\vec{F}$.

(b) Consider the following integral:

$$\int_0^2 \int_0^{3\sqrt{1-z^2/4}} \int_0^{2\sqrt{1-y^2/9-z^2/4}} xyz \, dxdydz$$

Sketch the region of integration and evaluate the integral.

(c) Find the flux of $\vec{F}$ through the surface bounding the region of integration in the previous part.

(a) $\nabla \cdot (x^2yz, xy^2z, xyz^2) = 2xyz + 2xyz + 2xyz = 6xyz$

(b) Region: Part of an ellipsoid centered at the origin in the region $x > 0, y > 0, z > 0$, with lengths 2,3,2 along x,y,z-axis.

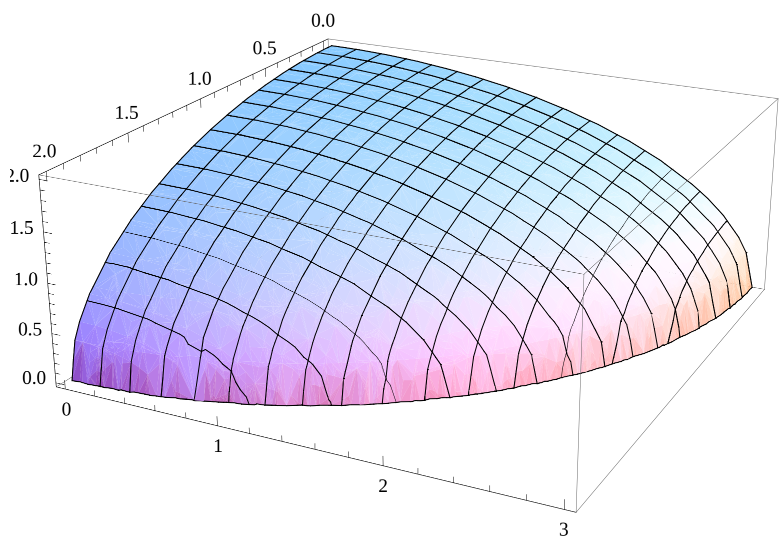

Figure 1: Region of Integration

$$\begin{aligned}
x &= 2u \\
y &= 3v \\
z &= 2w \\
dxdydz &= 12dudvdw \\
\int\int\int xyz\ dxdydz &= \int\int\int 12uvw\ (12dudvdw) \\
&= 144\int\int\int uvdudvdw \\
&= 144\int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \rho^3\cos(\theta)\sin(\phi)\sin(\theta)\sin(\phi)\cos(\phi)\rho^2\sin(\phi)d\rho d\theta d\phi \\
&= 144\int_0^1 \rho^5 d\rho \int_0^{\pi/2}\cos(\theta)\sin(\theta)d\theta\int_0^{\pi/2}\sin(\phi)^3\cos(\phi)d\phi \\
&= 144 * 1/6 * 1/2 * 1/4 = 3
\end{aligned}$$

(c) Use Divergence Theorem:

$$\begin{aligned}
\int_S \vec{F}\cdot d\vec{A} &= \int_W \nabla\cdot\vec{F}dV \\
&= 6\int_W xyzdV \\
&= 6 * 3 = 18
\end{aligned}$$