Notes on Hamiltonian formalism

① $(\mathcal{M}, \omega)$ - symplectic mfd

$\omega \in \Omega^2(\mathcal{M})$, $d\omega = 0$, non-degenerate

Local coordinates

$p_1 \cdots p_n, q^1 \cdots q^n$

$\omega = \sum dp_i \wedge dq^i$

<u>Example</u> $\mathcal{M} = T^* M$

$\omega = d\theta$

where $\theta(\xi) = \langle p, \pi_* \xi \rangle$

$\xi \in T_{p,q}(T^* M)$

$\pi : T^* M \longrightarrow M$

$\theta = \sum p_i dq^i$

ω gives $T\mathcal{M} \xrightarrow{\sim} T^*\mathcal{M}$

$X \mapsto \omega(-, X)$

$\partial_{q^i} \mapsto dp_i$

$\partial_{p_i} \mapsto -dq^i$

and thus

$C^\infty(\mathcal{M}) \longrightarrow Vect(\mathcal{M})$

$f \longmapsto X_f : \omega(-, X_f) = df$

$X_f = \sum \left(\frac{\partial f}{\partial p_i} \partial_{q^i} - \frac{\partial f}{\partial q^i} \partial_{p_i} \right)$

<u>Lemma</u> For $X \in Vect(\mathcal{M})$,

$\mathcal{L}_X \omega = 0 \iff$ locally, $X = X_f$

(2) Eqs of motion

Given $H \in C^\infty(\mathcal{M})$ (Hamiltonian), eqs of motion:

$$\dot{x} = X_H(x)$$

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}$$

(3) Poisson bracket

$$\{f, g\} = \partial_{X_g} f = \omega(X_g, X_f)$$

$$\{f,g\} = \cancel{\sum \frac{\partial g}{\partial q^i}\frac{\partial f}{\partial p_i} - \frac{\partial f}{\partial}} = \sum\left(\frac{\partial g}{\partial p_i}\frac{\partial f}{\partial q^i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}\right)$$

Thm (1) $\{\ \}$ is skew-symmetric and satisfies Jacobi identity

(2) $X_{\{f,g\}} = [X_f, X_g]$

where $[X, Y]$ is defined by

$$\partial_{[X,Y]}\varphi = (\partial_Y \partial_X - \partial_X \partial_Y)\varphi$$

(note order)

$$\{p_i, p_j\} = \{q^i, q^j\} = 0$$

$$\{q^i, p_j\} = \delta_{ij}$$

Eqs of motion using Poisson bracket:

$$\dot{f} = \partial_{X_H} f = \{f, H\}$$

So f is integral of motion $\Leftrightarrow \{f, H\} = 0$

In particular, H is an integral.

④ Noether's theorem and moment map

G - a Lie group

$\mathfrak{g} = T_1 G$ - Lie algebra

If G acts on $\mathcal{M}$, preserving ω,

so we have $\rho: G \to Sympl(\mathcal{M})$,

then $\rho_*: \mathfrak{g} \to Vect(\mathcal{M})$

$$\mathcal{L}_{\rho_* a}\, \omega = 0 \qquad a \in \mathfrak{g}$$

<u>Lemma</u> $\rho_*([a,b]) = [\rho_*(a), \rho_*(b)]$

<u>Def</u> ρ is called Hamiltonian if

$\exists$ $\mathfrak{g} \to C^\infty(\mathcal{M})$

$a \mapsto H_a$

s.t. ① It is linear

② $\rho_* a = X_{H_a}$

③ $\{H_a, H_b\} = H_{[a,b]}$

Equivalently, it means we have

$$\mu: \mathcal{M} \to \mathfrak{g}^*$$
$$x \to H_-(x)$$

called moment map

Thm Let $\mathcal{M} = T^*M$

$$\rho : G \to \mathrm{Diff}(M)$$

then lifting $\tilde{\rho} : G \to \mathrm{Sympl}(\mathcal{M})$ is Hamiltonian, with

$$H_a(p,q) = \langle \rho_*(a)(q), p \rangle$$

$$q \in M, \quad p \in T_q^* M, \quad a \in \mathfrak{g}$$

Example $M = \mathbb{R}^n$

$G = GL(n, \mathbb{R})$, acting on $\mathbb{R}^n$

(we consider vectors in $\mathbb{R}^n$ as column vectors)

$\mathfrak{g} = \mathfrak{gl}(n, \mathbb{R}) =$ all matrices

If $a = (a_{ij}) \in \mathfrak{gl}(n, \mathbb{R})$, then

$\rho_*(a) = \sum a_{ij}\, q^j\, \partial_{q^i} \in \mathrm{Vect}(\mathbb{R}^n)$

$H_a(p,q) = \sum a_{ij}\, q^j p_i =: M_a$

These are called angular momenta.

(5) Angular momenta

Special case of previous:

$G = SO(3,\mathbb{R})$, acting on $M = \mathbb{R}^3$

$\mathfrak{g} = so(3,\mathbb{R}) = \{a \in Mat_{3\times3}(\mathbb{R}) \mid a + a^t = 0\}$

$= \langle J_x, J_y, J_z \rangle$

$J_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, so that $\exp(t J_z) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}$

is angle t ~~to~~ counterclockwise rotation in xy plane

J_y, J_x are obtained from J_z by cyclic permutation of variables $x \to y \to z$

$J_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$, $J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$

Commutation relations:

$[J_x, J_y] = J_z$, $[J_y, J_z] = J_x$, $[J_z, J_x] = J_y$

Corresponding vector fields on $\mathbb{R}^3$

$\rho_* J_x = y\partial_z - z\partial_y$

$\rho_* J_y = z\partial_x - x\partial_z$

$\rho_* J_z = x\partial_y - y\partial_x$

Corresp. generating functions $H_a(p,q)$ are called angular momenta:

$M_x = y p_z - z p_y$

$M_y = z p_x - x p_z$

$M_z = x p_y - y p_x$

$\{M_x, M_y\} = M_z$, $\{M_y, M_z\} = M_x$, $\{M_z, M_x\} = M_y$

Note: in this case the situation is confusing since there is an isomorphism

$$\mathbb{R}^3 \simeq so(3)$$
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \longrightarrow x J_x + y J_y + z J_z$$

which commutes with action of $SO(3)$.

Under this isomorphism, $[\,,\,]$ in $\mathfrak{g}$ is identified with cross-product $\times$ so all of the above can be rewritten in terms of cross products. But we will not do this.

⑥ Integrable systems

<u>Def</u> A classical mechanical system $(\mathcal{M}, \omega, H)$ is called completely integrable (or Liouville integrable) if there exist $F_1 \dots F_n \in C^\infty(\mathcal{M})$ (where $\dim \mathcal{M} = 2n$) s.t.

① H can be written as function of $F_1 \dots F_n$:
$\exists\, \tilde{H} \in C^\infty(\mathbb{R}^n)$ s.t. $H(x) = \tilde{H}(F_1(x) \dots F_n(x))$

② $\{F_i, F_j\} = 0$

③ At generic point $x \in \mathcal{M}$, $dF_1 \dots dF_n \in T^*_x \mathcal{M}$ are linearly indep.

Note: this implies $\{F_i, H\} = 0$, so each F_i is an integral of motion

<u>Examples</u> ① For $n=1$, every system is integrable: take $F_1 = H$

② Particle in central field in $\mathbb{R}^3$ is integrable:

$$F_1 = H, \quad F_2 = M_z, \quad F_3 = M_x^2 + M_y^2 + M_z^2$$

Thm (Arnold – Liouville)

Let (M, ω, H) be an integrable system.
Denote $F : M \to \mathbb{R}^n$
$x \mapsto F_1(x) \dots F_n(x)$
and let $M_f = F^{-1}(f)$, $f \in \mathbb{R}^n$, be the level set.
Assume M_f is compact and connected. Then:

① $M_f \simeq T^n = \mathbb{R}^n / \mathbb{Z}^n$

② One can choose coordinates $I_1 \dots I_n$, $\varphi_1 \dots \varphi_n$ in neighborhood of M_f s.t.

- I_i only depend on $F_1 \dots F_n$
- $x \mapsto (\varphi_1(x) \dots \varphi_n(x))$ is isomorphism for all $M_{f'} \to T^n$ level sets $M_{f'}$ near M_f
- $\omega = \sum dI_i \wedge d\varphi_i$

These are called action-angle variables.

Corollary $\dot{I}_i = 0$, $\dot{\varphi}_i = \omega_i(f)$, so the motion is quasiperiodic in each fiber M_f.

Moreover, eqs of motion can be solved in quadratures.

Example $n = 1$, $M = \mathbb{R}^2$, $\omega = dp \wedge dq$, $H = \frac{1}{2}(p^2 + q^2)$
(harmonic oscillator).

If we denote by (r, φ) usual polar coordinates in $\mathbb{R}^2$, then $(I = \frac{1}{2} r^2, \varphi)$ are action-angle variables, and $\dot{\varphi} = 1$

(7) Rotation of solid body in $\mathbb{R}^n$

(a) For a Lie group G, we have actions on itself, $\mathfrak{g}$:

<u>Left</u>: $L_g : G \to G$ $\quad (L_g)_* : T_x G \to T_{gx} G$

$x \mapsto gx \quad (L_{g^{-1}})^* : T_x^* G \to T_{gx}^* G$

For simplicity, we will write just $g.v$, $g.\alpha$ instead of $(L_g)_* v$, $(L_{g^{-1}})^* \alpha$

<u>Right</u>: $R_g : G \to G$ $\quad (R_g)_* : T_x G \to T_{xg} G$

$x \mapsto xg \quad (R_{g^{-1}})^* : T_x^* G \to T_{xg}^* G$

Again, we will use notation vg, αg

<u>Adjoint</u> $\quad Ad_g : G \to G \qquad \mathfrak{g} \to \mathfrak{g} \qquad \mathfrak{g}^* \to \mathfrak{g}^*$

$x \mapsto gxg^{-1} \qquad a \to gag^{-1} \qquad \lambda \mapsto g\lambda g^{-1}$

Note: for $g = e^{ta}$,

$$\frac{d}{dt}\Big|_{t=0} Ad_g(b) = ad_a b = [a, b], \quad a, b \in \mathfrak{g}$$

We will denote by $[\,,]$ the coadjoint action on $\mathfrak{g}^*$:

$$[a, \lambda] = ad_a^* \lambda = \frac{d}{dt}\Big|_{t=0} g\lambda g^{-1}, \quad g = e^{ta}, \quad \lambda \in \mathfrak{g}^*$$

(b) Solid body in $\mathbb{R}^n$ - Euler's top

For any point in the body

$$\begin{pmatrix}\text{coordinates}\\ \text{at time } t\end{pmatrix} = g(t)\cdot\begin{pmatrix}\text{coordinates}\\ \text{at time } 0\end{pmatrix}$$

$$g(t) \in SO(n, \mathbb{R})$$

Coordinates are in fixed frame $e_1 \dots e_n$ in $\mathbb{R}^n$

One can also define moving frame

$e_i(t) = g(t)\, e_i$

and coordinates in this frame (which will be constant)

So configuration space is $M = G = SO(n, \mathbb{R})$

Velocity at time t: $\dot{g}(t) \in T_{g(t)}G$

Lemma
$$\dot{g}(t) = \omega \cdot g(t)$$
$$= g(t)\,\Omega$$

where $\omega \in \mathfrak{g}$ is instant velocity in fixed frame

$\Omega \in \mathfrak{g}$ is instant velocity in moving frame

$$\Omega = g^{-1}\cdot \omega \cdot g = g^{-1}\dot{g}(t)$$

Kinetic energy: $K = \frac{1}{2}\langle J\Omega, \Omega\rangle$

for some $J : \mathfrak{g} \to \mathfrak{g}^*$ - a symmetric pos. def form on $\mathfrak{g}$

Thus, K is a left-invariant metric on G

Trajectories are geodesics of this metric

(c) Hamiltonian formalism

Phase space: T^*G

Hamiltonian: $H(p,g) = \frac{1}{2}\langle J^{-1}M, M\rangle$

$p \in T_g^*G$, $M = g^{-1}\cdot p$ "momentum in moving frame"

This system is invariant under left action of G on itself. This action is Hamiltonian:

$H_a(p,g) = \langle p, (\rho_* a)(g)\rangle = \langle p, ag\rangle = \langle pg^{-1}, a\rangle$

so $m = pg^{-1}$ is an integral of motion: $\dot{m} = 0$

$$M = g^{-1} m g, \qquad \dot{g} = g\Omega \qquad \Omega \in \mathfrak{g}$$

$$\dot{M} = [\Omega, M]$$

So eqs of motion are

$$\begin{cases} \dot{M} = [\Omega, M] \\ \dot{g} = g\cdot\Omega \end{cases} \qquad M \in \mathfrak{g}^*, \ \Omega = J^{-1}M \in \mathfrak{g}$$

(Euler equations). In particular, M stays in the same coadjoint orbit.

If $K = \frac{1}{2}\langle J\Omega, \Omega\rangle$ is two-sided invariant, then $J: \mathfrak{g} \to \mathfrak{g}^*$ commutes with G-action, so $[\Omega, M] = 0$ and $\dot{M} = 0$, $\Omega = \text{const}$, and trajectories are $g(t) = e^{t\Omega}$

ⓓ Euler's top in $\mathbb{R}^3$ as integrable system

Integrals of motion:

- $H = \frac{1}{2} \langle J^{-1} M, M \rangle$
- $m = gMg^{-1}$

Components of m satisfy the commutation relations of $so(3)$:

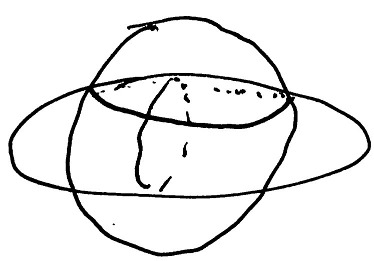
$$\{m_x, m_y\} = m_z \,, \quad \text{etc.}$$

Thus:

$$F_1 = H$$
$$F_2 = m_z$$
$$F_3 = M^2 = m_x^2 + m_y^2 + m_z^2 = M_x^2 + M_y^2 + M_z^2$$

are Poisson commuting integrals of motion

In particular, M belongs to intersection of sphere $M^2 = \text{const}$ and ellipsoid $\langle J^{-1} M, M \rangle = \text{const}$



Full motion is a quasiperiodic motion in T^3: rotation, precession, nutation