MAT 552: PROBLEM SET 5
DUE TUESDAY 11/16

INSTRUCTOR: ALEXANDER KIRILLOV

---

1. Let $\sigma = \exp(\frac{\pi}{2}(e-f)) \in \mathrm{SL}(2,\mathbb{C})$.

(a) Write explicitly σ as a 2×2 matrix.

(b) Write adjoint action of σ, σ^2 in the basis e, f, h.

(c) Write action of σ, σ^2 in each of irreducible representations V_k discussed in class, using the basis $v, v^1, v^2, \dots$ [Hint: use part (a) and isomorphism $V_k \simeq S^k\mathbb{C}^2$.]

2. Let V_k be the $(k+1)$-dimensional irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$ discussed in class.

(a) Show that if V is a finite-dimensional representation of $\mathfrak{sl}(2,\mathbb{C})$, then $V \simeq \bigoplus n_k V_k$, and $n_k = \dim V[k] - \dim V[k+2]$, where $V[k]$ is the subspace of vectors of weight k: $V[k] = \{v \in V \mid hv = kv\}$.

(b) Show that $\sum n_k = \dim V[0]$, and that $\dim V[k] = \dim V[-k]$.

(c) Show that $f^k \colon V[k] \to V[-k]$ is an isomorphism of vector spaces.

3. Show that any representation V of $\mathrm{SO}(3,\mathbb{R})$ can be written as the direct sum $V \simeq \bigoplus n_{2k} V_{2k}$, where V_{2k} is an irreducible $(2k+1)$-dimensional representation, and $n_{2k} = \dim V[2k] - V[2k+2]$, where $V[\lambda] = \{v \in V \mid J_z v = \frac{i\lambda}{2} v\}$.

The rest of this problem set is about the Laplace operator on the sphere, $\Delta_{sph} = J_x^2 + J_y^2 + J_z^2$, which was discussed in Problem Set 2. Recall that this operator acts in the space $C^\infty(S^2)$ and commutes with rotations. The goal of this problem set is to diagonalize it. To avoid technicalities related to convergence, we use the space P_n of complex functions on S^2 which can be written as polynomials in x, y, z of total degree $\le n$.

4. Show that the functions

$$f_{p,k} = z^p\left(\sqrt{1-z^2}\right)^{|k|} e^{ik\varphi}, \qquad p \in \mathbb{Z}_+, k \in \mathbb{Z}, p + |k| \le n$$

where φ is defined by $x = r\cos\varphi, y = r\sin\varphi$, is a basis of P_n.

5. Show that P_n is a representation of $\mathfrak{so}(3,\mathbb{R})$.

6. Show that each of the functions $f_{p,k}$ is an eigenfunction for J_z, and calculate the eigenvalue.

7. Show that as a representation of $\mathfrak{so}(3,\mathbb{R})$,

$$P_n \simeq V_0 \oplus V_2 \oplus \dots \oplus V_{2n}$$

where V_{2k} is $(2k+1)$-dimensional irreducible representation.

8. Calculate the eigenvalues and multiplicities of Δ_{sph} in P_n.