# MAT 552: PROBLEM SET 1
# DUE TUESDAY 9/14

INSTRUCTOR: ALEXANDER KIRILLOV

---

**1.** Let $G$ be a Lie group and $H$ — a Lie subgroup.
(a) Let $\overline{H}$ be the closure of $H$ in $G$. Show that $\overline{H}$ is a subgroup in $G$.
(b) Show that each coset $Hx, x \in \overline{H}$, is open and dense in $\overline{H}$.
(c) Show that $\overline{H} = H$, that is, every Lie subgroup is closed.

**2.** (a) Show that every discrete normal subgroup of a Lie group is central (hint: consider the map $G \to N\colon g \mapsto ghg^{-1}$ where $h$ is a fixed element in $N$).
(b) By applying part (a) to kernel of the map $\widetilde{G} \to G$, show that for any Lie group $G$, the fundamental group $\pi_1(G)$ is commutative.

**3.** Let $G_{n,k}$ be the set of all dimension $k$ subspaces in $\mathbb{R}^n$ (usually called the Grassmanian). Show that $G_{n,k}$ is a homogeneous space for the group $O(n)$ and thus can be identified with coset space $O(n)/H$ for appropriate $H$. Use it to prove that $G_{n,k}$ is a manifold and find its dimension.

**4.** Describe explicitly the tangent space $\mathfrak{sp}(4) \subset \mathfrak{gl}(4)$. Find dimension of $\mathrm{Sp}(4)$.

The next series of problems is about the group $\mathrm{SU}(2)$ and its adjoint representation, i.e. its action on the tangent space at identity: $T_e\mathrm{SU}(2) = \mathfrak{su}(2) = \{a \in \mathrm{Mat}(2 \times 2, \mathbb{C}) \mid a = -\overline{a}^t\}$ (which implies that $\mathfrak{su}(2)$ is a 3-dimensional real vector space). Recall that the adjoint action is given by $\mathrm{Ad}\, g\colon a \mapsto gag^{-1}$.

**5.** Define a bilinear form on $\mathfrak{su}(2)$ by $(a, b) = \mathrm{tr}(a\overline{b}^t)$. Show that this form is symmetric and positive definite.

**6.** Show that this form is invariant under the adjoint action of $\mathrm{SU}(2)$. Choose an orthonormal basis in $\mathfrak{su}(2)$ and write explicitly the matrix describing the action of $g \in \mathrm{SU}(2)$ in this basis. Show that this defines a Lie group homomorphism $\varphi\colon \mathrm{SU}(2) \to \mathrm{SO}(3)$.

**7.** Let $\mathfrak{so}(3) = T_e\mathrm{SO}(3)$. Let $\varphi_*\colon \mathfrak{su}(2) \to \mathfrak{so}(3)$ be the map of tangent spaces induced by $\varphi$. Show that $\varphi_*$ is an isomorphism.

**8.** Deduce from the previous problem that $\varphi$ is a diffeomorphism of a neighborhood of 1 in $\mathrm{SU}(2)$ to a neighborhood of 1 in $\mathrm{SO}(3)$. Show that $\ker \varphi$ is a discrete normal subgroup in $\mathrm{SU}(2)$, and that $\mathrm{Im}\, \varphi$ is an open subgroup in $\mathrm{SO}(3)$.

**9.** Prove that $\varphi$ establishes an isomorphism $\mathrm{SU}(2)/\mathbb{Z}_2 \to \mathrm{SO}(3)$ and thus, since $\mathrm{SU}(2) \simeq S^3$ (proved in class), $\mathrm{SO}(3) \simeq \mathbb{RP}^3$.