MAT 535: HOMEWORK 9

DUE WED, APRIL 16

In all problems, $\overline{K}$ stands for the algebraic closure of a field K, K^* stands for the multiplicative group of non-zero elements in K, and $\mathbb{F}_q$ stands for the finite field with $q = p^n$ elements.

1. (a) Let $K \subset L \subset M$ be a chain of algebraic extensions (not necessarily finite). Prove that any embedding $\varphi\colon L \to \overline{K}$ such that $\varphi|_K = \mathrm{id}$ can be extended to an embedding $M \to \overline{K}$.

(b) Prove that if $L_1, L_2 \subset \overline{K}$ are two subfields containing K such that there exists an isomorphism $\varphi\colon L_1 \to L_2$ with $\varphi|_K = \mathrm{id}$, then φ can be extended to an isomorphism $\overline{K} \to \overline{K}$.

2. Prove that the multiplicative group of a finite field is always cyclic (hint: use the last problem from the previous assignment).

3. Let η be a primitve p-th root of unity in $\mathbb{C}$. Show that if p is prime, $p > 2$, then the cyclotomic field $\mathbb{Q}(\eta)$ contains either $\sqrt{p}$ or $\sqrt{-p}$. [Hint: take a be the generator of the cyclic group $(\mathbb{Z}_p)^*$; show that the elements

$$\varepsilon_1 = \eta^a + \eta^{a^3} + \cdots + \eta^{a^{p-2}}; \qquad \varepsilon_2 = \eta^{a^2} + \eta^{a^4} + \cdots + \eta^{a^{p-1}}$$

satisfy a quadratic equation.]

4. Prove that any finite extension of a finite field is simple, i.e. is generated by a single element.

5. Show that for any $a \in \mathbb{F}_p^*$, the polynomial $x^p - x - a$ is irreducible over $\mathbb{F}_p$

6. Find the number of irreducible polynomials of degree 3 over $\mathbb{F}_3$; of degrees 2 and 3 over $\mathbb{F}_9$.

7. Let $\alpha, \beta \in \overline{K}$. Prove that the following are equivalent:

- Minimal polynomial of α over K is the same as minimal polynomial of β.
- There exists an automorphism $\varphi\colon \overline{K} \to \overline{K}$ such that $\varphi|_K = \mathrm{id}$, $\varphi(\alpha) = \beta$.