MAT 535: HOMEWORK 7

DUE WED, APRIL 2

Throughout this problem set, we use the folowing conventions:
– G is a finite group.
– all representations are **complex and finite-dimensional**
– V_i, $i \in Irr(G)$ — full list of all irreducible representations of G (up to isomorphism)
– $\chi_V(g) = \mathrm{tr}_V(g)$ — character of a representation V

1. Let V be a representation of G.
   (a) Prove that $\chi_{V^*} = \overline{\chi_V}$. (Hint: choose an orthonormal basis in V.)
   (b) Prove that $V \simeq V^*$ iff χ_V is real-valued.

2. This problem is about representations of the symmetric group S_4.
   (a) Consider the action of S_4 on $\mathbb{C}^4$ by permutation of components. Prove that then $\mathbb{C}^4 = \mathbb{C} \oplus V$, where $\mathbb{C}$ is the trivial one-dimensional representation of S_4, and V is an irreducible three–dimensional representation.
   (b) Compute the characters of $\mathbb{C}^4, V$ by explicitly writing their values on each conjugacy class in S_4 (you will need to recall what are conjugacy classes in symmetric group).
   (c) Find the number and dimensions of all irreps of S_4 (you are not required to give an explicit construction of the representations). [Hint: use the last problem of previous HW]

3. Consider the regular representation of G, i.e. a representation in functions on G. Compute the character of this representation and prove that its decomposition into irreducible representations is given by $V \simeq \bigoplus n_i V_i$, where $n_i = \dim V_i$. (Hint: you can use $n_i = (\chi_V, \chi_{V_i})$.)

4. Let V be a representation of G, and χ_V — its character. Prove that then

$$\frac{1}{|G|} \sum_G \chi_V(g) = \dim V^G$$

5. Let $K \subset L \subset M$ be a chain of field extensions. Prove that $M{:}K$ is a finite extension iff both $M{:}L$, $L{:}K$ are finite and that if they are finite, then $[M{:}K] = [M{:}L] \cdot [L{:}K]$.

6. For each of the following complex numbers, determine its degree and minimal polynomial over $\mathbb{Q}$.

(a) $1 + i$ (b) $1 - i\sqrt{2}$

(c) $1 + \sqrt[3]{5}$ (d) $\sqrt{1 + \sqrt{2}}$

7. Let $K = \mathbb{Q}(\sqrt{2} + \sqrt{3})$
   (a) Prove that $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ (hint: $\sqrt{6} \in K$)
   (b) Prove that $[K : \mathbb{Q}] = 4$.
   (c) Find the minimal polynomial of $\sqrt{2} + \sqrt{3}$ over $\mathbb{Q}$.