# MAT 535: HOMEWORK 6

DUE WED, MAR 12

Throughout this problem set, all representations are finite-dimensional.

**1.** Classify all irreducible complex representations of the dihedral group

$$D_4 = \langle a, b \mid a^4 = b^2 = 1, \quad bab^{-1} = a^{-1} \rangle$$

[Hint: $D_4$ contains $\mathbb{Z}_4$, so any such representation is also a representation (not necessarily irreducible) of $\mathbb{Z}_4$. ]

**2.** For a representation $V$ of a group $G$, define the subspace of $G$-invariants by

$$V^G = \{v \in V \mid gv = v \quad \forall g \in G\}$$

(a) Show that $V \mapsto V^G$ is a functor from the category of representations of $G$ to vector spaces.

(b) Show that this functor is left exact: if $0 \to A \to B \to C \to 0$ is a short exact sequence of representations of $G$, then $0 \to A^G \to B^G \to C^G$ is an exact sequence of vector spaces.

(c) Show that in general, this functor is not exact: in the notation of the previous part, $0 \to A^G \to B^G \to C^G \to 0$ may fail to be exact at the last term. [Hint: take $B$ to be two-dimensional representation of $\mathbb{Z}$ given by $n \mapsto \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$. ]

(d) Show that if $G$ is finite, then this functor is exact.

**3.** Let $V, W$ be irreducible representations of a group $G$ over an algebraically closed field. As was discussed in class, this automatically gives a structure of representation on $V \otimes W$. Prove that $(V \otimes W)^G$ is one-dimensional if $V$ is isomorphic to $W^*$ and zero otherwise.

**4.** Let $C$ be the standard cube in $\mathbb{R}^3$: $C = \{|x_i| \le 1\}$, and let $S$ be the set of faces of $C$ (thus, $S$ consists of 6 elements). Consider the 6-dimensional complex vector space $V$ of functions on $S$, and define $A\colon V \to V$ by

$$(Af)(\sigma) = \frac{1}{4} \sum_{\sigma'} f(\sigma')$$

where the sum is taken over all faces $\sigma'$ which are neighbors of $\sigma$ (i.e., have a common edge with $\sigma$). The goal of this problem is to diagonalize $A$.

(a) Let $G = \{g \in GL(3, \mathbb{R}) \mid g(C) = C\}$ be the group of symmetries of $C$. Show that $A$ commutes with the natural action of $G$ on $V$.

(b) Let $z = diag(-1, -1, -1) \in G$ be the diagonal matrix with $-1$ on the diagonal. Show that as a representation of $G$, $V$ can be decomposed in the direct sum

$$V = V_+ \oplus V_-, \qquad V_\pm = \{f \in V \mid zf = \pm f\}.$$

(c) Show that as a representation of $G$, $V_+$ can be decomposed in the direct sum

$$V_+ = V_+^0 \oplus V_+^1, \quad V_+^0 = \{f \in V_+ \mid \sum_\sigma f(\sigma) = 0\}, \quad V_+^1 = \mathbb{C} \cdot 1$$

where 1 denotes the constant function on $S$ whose value at every $\sigma \in S$ is 1.

(d) Find the eigenvalues of $A$ on $V_-, V_+^0, V_+^1$.

[Note: in fact, each of $V_-, V_+^0, V_+^1$ is an irreducible representation of $G$, but you do not need this fact.]

**5.** Let $G$ be a finite group, and let $\mathbb{F}$ be algebraically closed; denote $Z =$ center of $\mathbb{F}[G] = \{f \in \mathbb{F}[G] \mid fx = xf \quad \forall x \in \mathbb{F}[G]\}$. Prove that

$$\begin{aligned} \dim Z &= \text{number of isomorphism classes of irreducible representations of } G \\ &= \text{number of conjugacy classes in } G \end{aligned}$$