MAT 535: HOMEWORK 5

DUE WED, MAR 5

Unless stated otherwise, $\mathbb{F}$ is an arbitrary field, R is a ring with unit.

1. Recall that for an R-module X we defined

$$\mathrm{Ext}^1(X, A) = H^1(0 \to \mathrm{Hom}_R(P_0, A) \to \mathrm{Hom}_R(P_1, A) \to \mathrm{Hom}_R(P_2, A) \to \cdots) \tag{1}$$

where $\cdots P_2 \to P_1 \to P_0 \to X \to 0$ is a projective resolution of C.

Prove that for any short exact sequence of R-modules $0 \to A \to B \to C \to 0$, one has the following exact sequence:

$$0 \to \mathrm{Hom}(X, A) \to \mathrm{Hom}(X, B) \to \mathrm{Hom}(X, C) \to \mathrm{Ext}^1(X, A)$$

[Hint: apply the result about long exact sequence of cohomology to the following short exact sequence of complexes $0 \to \mathrm{Hom}(P^\bullet, A) \to \mathrm{Hom}(P^\bullet, B) \to \mathrm{Hom}(P^\bullet, C) \to 0$.]

2. Prove that the matrix algebra $\mathrm{Mat}_n(\mathbb{F})$ is simple, i.e. has no non-trivial two sided ideals. [Hint: every such ideal would contain a matrix unit E_{ij}.]

3. Prove the following version of Schur's lemma: if $\mathbb{F}$ is algebraically closed (i.e., every polynomial $p \in \mathbb{F}[x]$ has a root in $\mathbb{F}$), R is an algebra over $\mathbb{F}$, and E — an R-module which is finite-dimensional as a vector space over $\mathbb{F}$, then $\mathrm{End}_R(E) = \mathbb{F}$. [Hint: if $f \in \mathrm{End}_R(E)$, then by Schur's lemma, for any $\lambda \in \mathbb{F}$, $f - \lambda$ is zero or invertible.]

4. Prove that a commutative ring is semisimple iff it is direct sum of fields.

5. (Some of you probably saw this problem before :)

A linear operator A in a finite-dimensional vector space V over $\mathbb{F}$ is called *semisimple* if V is semisimple as a module over $\mathbb{F}[A]$.

(a) Prove that if A is semisimple, and $W \subset V$ is A-invariant, then restriction of A to W is also semisimple.

(b) Prove that for an algebraically closed field $\mathbb{F}$, A is semisimple if and only if A is diagonalizable.

6. Let $N \subset R$ be the intersection of all *maximal* left ideals L in R. Prove that N is a two sided ideal, and that N acts by zero in every simple R-module E. Deduce from this that if R is semisimple, then $N = 0$.