MAT 535: HOMEWORK 4

DUE WED, FEB 27

Unless stated otherwise, $\mathbb{F}$ is an arbitrary field, R is a ring with unit, and $\mathcal{C}$ is the category of modules over R (in fact, almost all results hold in greater generality, for any abelian category, but you are not required to show that).

1. Let $A^\bullet = (0 \to A_1 \to A_2 \to A_3 \to 0)$, $B^\bullet = (0 \to B_1 \to B_2 \to B_3 \to 0)$ be two short exact sequences, and $f\colon A^\bullet \to B^\bullet$ a chain map such that $f_1\colon A_1 \to B_1$, $f_3\colon A_3 \to B_3$ are isomorphisms. Prove that $f_2\colon A_2 \to B_2$ is also an isomorphism.

2. Let P_1, P_2 be two R-modules. Prove that $P_1 \oplus P_2$ is projective iff both P_1, P_2 are projective.

3. (a) Prove that any R-module M has a *projective resolution*, i.e. there is an exact sequence

$$\dots P_2 \to P_1 \to P_0 \to M \to 0$$

where P_i is projective.

(b) Prove that any two such projective resolutions are chain homotopic: if $P_\bullet \to M \to 0$, $Q_\bullet \to M \to 0$ are two projective resolutions, then there exists a chain map $f\colon P_\bullet \to Q_\bullet$ such that f is identity on M. Moreover, such f is unique up to homotopy: if f_1, f_2 are two such maps then they are chain homotopic (see problem 4 from previous HW for definition).

(c) Prove that if R is a PID, then any module has a projective resolution of length 2, i.e. a projective resolution $0 \to P_1 \to P_0 \to M \to 0$.

4. (a) Compute $\mathrm{Ext}^1(\mathbb{Z}_2, \mathbb{Z})$ (in the category of $\mathbb{Z}$-modules).

(b) Let $R = \mathbb{F}[x]$; for any $\lambda \in \mathbb{F}$, define $M_\lambda = R/(x-\lambda)$ (this is one-dimensional vector space over $\mathbb{F}$ in which x acts by λ). Compute $\mathrm{Ext}^1(M_\lambda, M_\mu)$.

5. (a) Prove that functor of tensor product is right exact, but not necessarily exact: if $0 \to A \to B \to C \to 0$ is a short exact sequence of R-modules, and M is an (S,R)-bimodule, then $M \otimes_R A \to M \otimes_R B \to M \otimes_R C \to 0$ is exact, but $0 \to M \otimes_R A \to M \otimes_R B \to M \otimes_R C \to 0$ is not necessarily exact at first term.

(b) Show that if M is free as a right R-module, then the functor of tensoring with M is exact.

6. This problem explains how, given an element $\varphi \in \mathrm{Ext}^1(C, A)$, one constructs the corresponding extension.

Let A, C be R-modules, $C = P_0/P_1$, and $\varphi \in \mathrm{Hom}_R(P_1, A)$. Define $B = (A \oplus P_0)/\{\varphi(x), x\}_{x \in P_1}$. Prove that then one has a short exact sequence $0 \to A \to B \to C \to 0$.

***7.** (This is a more complicated problem; you can do it for extra credit. It can be turned in later.)

Recall that we have given two definitions of $\mathrm{Ext}^1(C, A)$ in class:

- Simple definition, only discussed in very special case when R is an algebra over a field $\mathbb{F}$:

$$\mathrm{Ext}^1(C,A) = \{f\colon R\times C \to A \mid r_1 f(r,c) = f(r_1 r, c) - f(r_1, rc)\}/\{f = r\varphi(c) - \varphi(rc),\ \varphi \in \mathrm{Hom}_{\mathbb{F}}(C,A)\} \tag{1}$$

- More general definition: if $\cdots P_2 \to P_1 \to P_0 \to C \to 0$ is a projective resolution of C, then

$$\mathrm{Ext}^1(C,A) = H^1(0 \to \mathrm{Hom}_R(P_0, A) \to \mathrm{Hom}_R(P_1, A) \to \mathrm{Hom}_R(P_2, A) \to \cdots) \tag{2}$$

The goal of this problem is to show that these two definitions are equivalent. Namely, let R be an algebra over $\mathbb{F}$.

(a) Define a complex (all tensor products are over $\mathbb{F}$):

$$R\otimes R\otimes R\otimes C \to R\otimes R\otimes C \to R\otimes C \to C \to 0$$

with differential defined by

$$d(r\otimes c) = rc$$
$$d(r_1\otimes r_2\otimes c) = (r_1 r_2)\otimes c - r_1 \otimes (r_2 c)$$
$$d(r_1\otimes r_2\otimes r_3\otimes c) = (r_1 r_2)\otimes r_3\otimes c - r_1\otimes (r_2 r_3)\otimes c + r_1\otimes r_2\otimes (r_3 c)$$

Prove that this is a projective resolution of C.

(b) Prove that computing $\mathrm{Ext}^1(C, A)$ using formula (2) and this projective resolution gives formula (1). (Hint: the map $\varphi \mapsto \varphi(1, x)$ gives an isomorphism $\mathrm{Hom}_R(R\otimes_F X, A) \simeq \mathrm{Hom}_{\mathbb{F}}(X, A)$.)