MAT 535: HOMEWORK 3

DUE WED, FEB 20

For a category $\mathcal{C}$, we denote by $\operatorname{Obj}\mathcal{C}$ the collection of objects of $\mathcal{C}$, and by $\operatorname{Mor}_{\mathcal{C}}(X,Y)$ the set of morphisms from object X to object Y.

1. Let Ab be the category of abelian groups. Define $D\colon \mathrm{Ab} \to \mathrm{Ab}$ by $D(G) = \operatorname{Hom}_{\mathbb{Z}}(G, \mathbb{Z})$.
   (a) Prove that D is a contravariant functor.
   (b) Construct a morphism of functors $G \to D(D(G))$.
   (c) Show that when restricted to the category of finitely generated abelian groups, the morphism $G \to D(D(G))$ becomes an isomorphism. [Hint: you will need the classification theorem for f.g. abelian groups for this.]

2. An object X in a category $\mathcal{C}$ is called *universal initial object* if for any object $Y \in \operatorname{Obj}\mathcal{C}$, there is a unique morphism $X \to Y$.
   (a) Show that universal object, if exists, is unique up to isomorphism: if X, X' are two universal objects, then there is an isomorpshim $X \simeq X'$.
   (b) Find the universal object in the category of vector spaces.
   (c) Let S be a finite set. Consider the category Gr_S whose objects are pairs (G, i), where G is a group and i is a map $S \to G$. Morphisms in this category are defined to be group homomoprhisms which agree with i:

$$\operatorname{Mor}((G_1, i_1), (G_2, i_2)) = \{f\colon G_1 \to G_2 \mid f \circ i_1 = i_2\}$$

   Prove that the universal object in this category is the free group generated by S.
   (d) Let $f\colon V \to W$ be a linear map between vector spaces V, W. Define a new category $\mathcal{C}_f$ (which depends on V, W, f) whose objects are pairs (X, g), where X is a vector space and $g\colon W \to X$ is a linear map such that $g \circ f = 0$. Morphisms in this category are linear maps which agree with g (similar to the previous part). Describe the universal object in this category.

3. For a fixed finite-dimensional vector space V, construct functorial isomorphisms $\operatorname{Hom}(V, -) \to - \otimes V^*$. (Long explanation: let functors $F_1, F_2\colon \mathcal{V}ect \to \mathcal{V}ect$, where $\mathcal{V}ect$ is the category of vector spaces over fixed field $\mathbb{F}$, be defined by $F_1(W) = \operatorname{Hom}(V, W)$, $F_2(W) = W \otimes V^*$. Construct isomorphism of functors $F_1 \to F_2$.)

4. Let $A^\bullet, B^\bullet$ be cochain complexes. Two cochain maps $f_1, f_2\colon A^\bullet \to B^\bullet$ are called *homotopic* if there is a collection of maps $h_n\colon A^n \to B^{n-1}$ such that $f_1 - f_2 = h \circ d_A + d_B \circ h$. Prove that in this case, f_1 and f_2 define the same map of cohomology spaces $H^n(A^\bullet) \to H^n(B^\bullet)$.

5. Let $A^\bullet$ be a finite cochain complex of finite-dimensional vector spaces. Define its Euler characteristic by

$$\chi(A^\bullet) = \sum_i (-1)^i \dim A^i$$

Prove that then

$$\chi(A^\bullet) = \sum_i (-1)^i \dim H^i(A^\bullet)$$

[Hint: $\dim(A^i/\ker d^i) = \dim \operatorname{Im} d^i$.]