MAT 535: HOMEWORK 2 (CORRECTED)

DUE WED, FEB 13

$\mathbb{F}$ stands for an arbitrary field of characteristic 0. Unless specified otherwise, V is a finite-dimensional vector space over $\mathbb{F}$.

1. Let V an n-dimensional vector space over V.

(a) Let $t \in V^{\otimes 4}$ be a tensor such that

$$s_{12}(t) = t, \qquad s_{34}(t) = -t \tag{1}$$

where $s_{12}(v_1 \otimes v_2 \otimes v_3 \otimes v_4) = v_2 \otimes v_1 \otimes v_3 \otimes v_4$, $s_{34}(v_1 \otimes v_2 \otimes v_3 \otimes v_4) = v_1 \otimes v_2 \otimes v_4 \otimes v_3$. Prove that then $\mathrm{Sym}(t) = \mathrm{Alt}(t) = 0$, where Sym and Alt are projectors $V^{\otimes 4} \to S^4 V$, $V^{\otimes 4} \to \Lambda^4 V$ respectively.

(b) Compute the dimension of the space of tensors satisfying condition (1)

2. Let $B\colon V \times V \to \mathbb{F}$ be a non-degenerate skew-symmetric bilinear form. Show that then there exists a basis in which B has the matrix

$$\begin{pmatrix} 0 & 1 & & & & & \\ -1 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & -1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 1 \\ & & & & & -1 & 0 \end{pmatrix}$$

(entries which are not shown are zeros). In particular, V must be even-dimensional.

3. Let $A\colon V \to V$ be a linear operator. Prove that then the characteristic polynomial can be written as follows:

$$p_A(x) = \det(x \cdot 1 - A) = \sum_k x^{n-k} (-1)^k \operatorname{tr}(\Lambda^k A)$$

where $n = \dim V$ and $\Lambda^k A\colon \Lambda^k V \to \Lambda^k V$ is the linear operator induced by A. [Hint: if $v_1, \dots, v_n$ is a basis in V, then $(x \cdot 1 - A)v_1 \wedge \dots \wedge (x \cdot 1 - A)v_n = \det(x \cdot 1 - A) v_1 \wedge \dots \wedge v_n$.]

4. (a) Let V be a finite-dimensional vector space over $\mathbb{F}$ with a skew-symmetric bilinear form b. Prove that then the formula

$$B(v_1 \wedge v_2, v_3 \wedge v_4) = b(v_1, v_3) b(v_2, v_4) - b(v_1, v_4) b(v_2, v_3) \tag{2}$$

defines on $\Lambda^2 V$ a symmetric bilinear form.

(b) Show that the space $\mathbb{C}^3$ with the standard symmetric bilinear form $\sum x_i y_i$ can be identified with the space $\Lambda^2(\mathbb{C}^2)$, with the bilinear form given by (2) for suitable choice of b. Does the same work over $\mathbb{R}$? (If we think of $\mathbb{C}^3$ as complexification of our usual 3-dimensional space, then elements of the space $\mathbb{C}^2$ are usually called *spinors* in physical literature.)

(c) Construct a group homomorphism $SL(2, \mathbb{C}) \to SO(3, \mathbb{C})$. [In fact, it is known that the kernel of this homomorphism is the subgroup $\{\pm 1\}$.]

5. Let $\xi \in \Lambda^p V$, $v \in V$, $v \neq 0$. Prove that $\xi \wedge v = 0$ iff $\xi = v \wedge \eta$ for some $\eta \in \Lambda^{p-1} V$.