MAT 535: HOMEWORK 12

DUE WED, MAY 7

In all problems, the fields are of zero characteristic. Notation ζ_n stands for a primitive n-th root of unity.

1. Show that $\cos(2\pi/5) = (\zeta_5 + \zeta_5^{-1})/2$ satisfies a quadratic equation over $\mathbb{Q}$. Write explicitly a formula for it, and explain how one can construct $\cos(2\pi/5)$ (and thus, a regular pentagon) using a straightedge and compass.

2. Let $x = \cos 20°$. Compute the minimal polynomial of x over $\mathbb{Q}$ (note: you need to prove that it is irreducible!) and compute its Galois group. [Hint: $\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$.] Prove that angle $60°$ can not be trisected by a straightedge and compass.

3. Which of the following angles can be constructed using a straightedge and compass?
- (a) $9°$
- (b) $10°$
- (c) $27°$

4. Prove that primitive roots of unity of order n form a basis of the cyclotomic filed $\mathbb{Q}(\zeta_n)$ over $\mathbb{Q}$ iff n is squarefree (i.e., is not divisible by a square of any prime number).

5. (Apollonius circles)
- (a) Prove that a circle with center at (x, y) and radius r is tangent to a given circle C iff x, y, r satisfy a quadratic equation of the form
$$x^2 + y^2 - r^2 + l(x, y, r) = 0$$
where l is a polynomial of degree one in x, y, r (coefficients of l depend on C)
- (b) Prove that given three circles C_1, C_2, C_3, constructing a circle which is tangent to all of them (such a circle is called the Apollonius circle) is equivalent to solving a system of three quadratic equations in three variables, which can be reduced to a system of two linear equations and a quadratic equation.
- (c) Prove that if C_1, C_2, C_3 are such that there exists a circle which is tangent to all of them, then such a circle can be constructed using a straightedge and compass.