# MAT 535: HOMEWORK 11

DUE WED, APRIL 30

In all problems, $\mathrm{Gal}(L/K)$ denotes the Galois group of $L$ over $K$.

**1.** Let $k$ be an arbitrary field, $x_1, \dots, x_n$ be independent variables and let $s_i \in k[x_1, \dots, x_n]$ be elementary symmetric polynomilas defined by

$$\prod(x - x_i) = x^n - s_1 x^{n-1} + \cdots + (-1)^n s_n \in k[x, x_1, \dots, x_n]$$

(a) Show that $s_i$ are algebraically independent; there is no non-trivial polynomial $p$ such that $p(s_1, \dots, s_n) = 0$. (hint: use induction in $n$)

(b) Recall that we have proved in class that $k(x_1, \dots, x_n)^{S_n} = k(s_1, \dots, s_n)$. Prove analogous statement for polynomials:

$$k[x_1, \dots, x_n]^{S_n} = k[s_1, \dots, s_n]$$

[ There are several ways to prove this. One way — normally given in textbooks — is to use induction in degree of a polynomial, using lexicographic ordering of monomials. However, there is a shorter proof, using the statement we had proved about symmetric rational functions and the fact that the ring of polynomials in several variables is a UFD.]

(c) Let $n = 3$; write the following polynomials as polynomials in $s_1, \dots, s_n$:
$x_1^2 + x_2^2 + x_3^2$
$x_1^2 x_2 + x_2^2 x_1 + x_1^2 x_3 + x_3^2 x_1 + x_2^2 x_3 + x_3^2 x_2$

**2.** Let $L$ be the splitting field of a polynomial $f(x)$ over $k$, and let $x_1, \dots, x_n \in L$ be all roots of this polynomial, so that $f(x) = c(x - x_1) \dots (x - x_n)$. Assume that $L$ is separable and thus Galois (this would hold automatically in characteristic zero) and let $G = \mathrm{Gal}(L/k)$.

(a) Let $D = \prod_{i<j}(x_i - x_j)^2$. Prove that $D \in L^G = k$.

(b) Prove that $k \subset k(\sqrt{D}) \subset L$ and $\mathrm{Gal}(L/k(\sqrt{D})) \subset A_n$ (the alternating group).

(c) Prove that if $f$ is an irreducible cubic polynomial, then:
- if $D$ is a square in $k$, then $[L : k] = 3$, $G = \mathbb{Z}_3$.
- if $D$ is not a square in $k$, then $[L : k] = 6$, $G = S_3$.

(It can be shown that up to a constant, $D$ coincides with the resultant $R(f, f')$ discussed in one of the previous homeworks.)

**3.** Let $f(x) = x^3 + px + q$. Prove that in this case, $D = -4p^3 - 27q^2$ (hint: compare $D$ with $f'(x_1)f'(x_2)f'(x_3)$, where $x_i$ are the roots.) Use it to decribe the Galois group of polynomials $x^3 - 3x + 1$; $x^3 - 3x + 3$ over $\mathbb{Q}$.

**4.** Compute the following Galois groups:

(a) Of the polynomial $x^n - t$ over $k = \mathbb{C}(t)$, where $t$ is an independent variable.

(b) Of the polynomial $x^3 - 10$ over $\mathbb{Q}$; over $\mathbb{Q}(\sqrt{-3})$.

**5.** Consider the cyclotomic field $\mathbb{Q}(\zeta)$ where $\zeta$ is the primitive root of 1 of order 15.

(a) Describe explicitly the Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$. Is it cyclic?

(b) Construct a tower of subfields

$$k_0 = \mathbb{Q} \subset k_1 \subset \cdots \subset k_n = \mathbb{Q}(\zeta)$$

so that $[k_{i+1} : k_i] = 2$.

(c) Show that one can write a formula for $\zeta$ which only involves rational numbers, arithmetic operations, and square roots.