MAT 535: HOMEWORK 10

DUE WED, APRIL 23

In all problems, $\overline{K}$ stands for the algebraic closure of a field K, K^* stands for the multiplicative group of non-zero elements in K, and $\mathbb{F}_q$ stands for the finite field with $q = p^n$ elements. $\mathrm{Gal}(L/K)$ denotes the Galois group of L over K.

1. (a) Prove that $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) = \mathbb{Z}_n$, with the generator F (Frobenius automorphism).
(b) Describe $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_{p^m})$, $n = dm$.

2. Let l be the splitting field of the polynomial $f(x) = x^3 - 5$ over $\mathbb{Q}$.
(a) Prove that $\mathrm{Gal}(L/\mathbb{Q}) = S_3$.
(b) Describe the fixed field L^{A_3}, where $A_3 \subset S_3$ is the alternating group.

3. Let $f(x) = x^4 - 2$
(a) Prove that f is irreducible over $\mathbb{Q}$
(b) Describe the splitting field L of f over $\mathbb{Q}$.
(c) Describe the Galois group $G = \mathrm{Gal}(L/\mathbb{Q})$
(d) Describe all possible subgroups in G. Which of them are normal?
(e) Describe all possible intermediate extensions $\mathbb{Q} \subset E \subset L$. Which of them are normal?

4. Describe the Galois group over $\mathbb{Q}$ of the following fields:
(a) Splitting field of $x^4 - 14x^2 + 9$
(b) $\mathbb{Q}(\sqrt{2 + \sqrt{2}})$ (hint: prove that it is isomorphic to $\mathbb{Z}_4$)