

**MAT 535: HOMEWORK 1**  
DUE WED, FEB 6

Problems marked by asterisk (\*) are optional.  $\mathbb{F}$  stands for an arbitrary field.

1. Compute the following tensor products of modules:
  - (a)  $\mathbb{Z}_n \otimes_{\mathbb{Z}} \mathbb{R}$
  - (b)  $\mathbb{Z}_n \otimes_{\mathbb{Z}} \mathbb{Z}_m$
  - (c)  $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$
2. Prove carefully that  $\mathbb{F}[x] \otimes_{\mathbb{F}} \mathbb{F}[y] \simeq \mathbb{F}[x, y]$ .
3. For a vector space  $V$  over  $\mathbb{R}$ , denote  $V_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} V$  (this is called *complexification* of a vector space). Conversely, for complex vector space  $W$  denote by  $W_{\mathbb{R}}$  the same set but considered as a vector space over  $\mathbb{R}$  (of dimension  $2 \dim_{\mathbb{C}} W$ ).
 

Describe, for a finite-dimensional complex vector space  $W$ , what is  $(W_{\mathbb{R}})_{\mathbb{C}}$ . Is it the same as the original space  $W$ ?
4. Let  $G$  be a finite group,  $H \subset G$  — a subgroup. Recall that  $\mathbb{F}[G]$  stands for the group algebra: formal linear combinations of elements of  $G$  with coefficients from  $\mathbb{F}$ .
 

Let  $R = \mathbb{F}[G] \otimes_{\mathbb{F}[H]} \mathbb{F}$ , where  $\mathbb{F}$  is considered as an  $\mathbb{F}[H]$ -module by letting  $(\sum f_i h_i)a = (\sum f_i)a$ ,  $f_i, a \in \mathbb{F}$ ,  $h_i \in H$  (in other words, every  $h \in H$  acts in  $\mathbb{F}$  by 1).

  - (a) Choose in each coset  $xH$  a representative  $g_i$ ,  $i \in G/H$ . Prove that then every element  $r \in R$  can be uniquely written as  $\sum_{i \in G/H} f_i g_i$ ,  $f_i \in \mathbb{F}$ . [Hint:  $\mathbb{F}[G]$  is free as a right  $\mathbb{F}[H]$ -module.]
  - (b) Show that  $\dim R = |G/H|$ .
  - (c) Show that  $R$  can be identified with the space of functions on  $G$  which are invariant under right multiplication by  $H$ :  $f(gh) = f(g)$  for any  $g \in G$ ,  $h \in H$ .

(This construction is called the *induced representation* and is very useful in representation theory.)
5. Let  $V$  be a finite-dimensional vector space over  $\mathbb{F}$ , with basis  $v_i$ . Let  $V^*$  be the dual vector space, with the dual basis  $v^j$ :  $v^j(v_i) = \delta_{ij}$ . Recall that it was discussed in class that one can identify  $\text{End}(V) = V \otimes_{\mathbb{F}} V^*$ .
  - (a) Show that under this identification the element  $\sum_i v_i \otimes v^i$  is identified with the identity operator  $1 \in \text{End } V$  (which, in particular, shows that it is independent of the choice of basis).
  - (b) Show that under this identification, the map  $\text{End } V \rightarrow \mathbb{F}: A \mapsto \text{tr}(A)$  is identified with the “evaluation map”:  $ev: V \otimes V^* \rightarrow \mathbb{F}: v \otimes f \mapsto f(v)$ .
6. Let  $V$ ,  $v_i$ ,  $v^i$  be as in the previous problem. Let  $B: V \times V \rightarrow \mathbb{F}$  be a symmetric bilinear form, and let  $\mu: V \times V \rightarrow V$  be a bilinear map. Define  $\tilde{\mu}: V \times V \times V \rightarrow \mathbb{F}$  by  $\tilde{\mu}(x, y, z) = B(\mu(x, y), z)$ .
  - (a) Show that  $B$  can be considered as an element in  $V^* \otimes V^*$ ,  $\mu$  as an element in  $V^* \otimes V^* \otimes V$ , and  $\tilde{\mu}$  as an element in  $V^* \otimes V^* \otimes V^*$  (all tensor products are over  $\mathbb{F}$ ).
  - (b) Show that  $\tilde{\mu} = 1 \otimes 1 \otimes ev \otimes 1(\mu \otimes B)$ , where  $1 \otimes 1 \otimes ev \otimes 1: V^* \otimes V^* \otimes V \otimes V^* \otimes V^* \rightarrow V^* \otimes V^* \otimes V^*$  is the evaluation map from the previous problem applied to the 3rd and 4th factors.
  - (c) Show that if one writes  $B, \mu, \tilde{\mu}$  in bases  $v_i, v^j$  as  $B = \sum b_{ij} v^i \otimes v^j$ ,  $\mu = \sum \mu_{ij}^k v^i \otimes v^j \otimes v_k$ , then  $\tilde{\mu} = \sum \tilde{\mu}_{ijl} v^i \otimes v^j \otimes v^l$ , with  $\tilde{\mu}_{ijl} = \sum \mu_{ij}^k b_{kl}$ .