MAT 535: MIDTERM

WED, MARCH 12

Your name:____

(please print)

	1	2	3	4	5	6	**Total**
Grade							

This is a take home exam; it should be returned on Friday, March 14, in class. Late submisssions **will not be accepted**.

You are allowed to quote any result from class, from homeworks, or from one of our standard textbooks (Dummit and Foote, Knapp, Lang). If you are using anything else, you are required to give a proof. The work must be your own: no discussions with anyone are allowed.

1. Let V be a finite-dimensional vector space over a field $\mathbb{F}$ of characteristic 0 and let $n = \dim V$. Choose an identification $\Lambda^n V \simeq \mathbb{F}$. Define, for every $0 \le k \le n$, a bilinear form

$$\Lambda^k V \otimes \Lambda^{n-k} V \to \Lambda^n V \simeq \mathbb{F}$$
$$\xi \otimes \eta \mapsto \xi \wedge \eta$$

(a) Prove that this form is non-degenerate, i.e. gives an identification $\Lambda^{n-k} V \simeq (\Lambda^k V)^*$.

(b) Assume $\mathbb{F} = \mathbb{R}$, $n = 4m$. Show that this form on $\Lambda^{2m} V$ is symmetric and compute the signature. (Recall that every non-degenerate symmetric form in a real vector space W in a suitable basis can be written as $\sum_{i=1}^{a} x_i^2 - \sum_{i=a+1}^{\dim W} \sum x_i^2$; the pair $(a, \dim W - a)$ is called the signature of the form.)

2. Let R, S be rings with unit, and R-mod, S-mod — categories of (left) modules over R, S respectively.

(a) Give an example of a functor $F\colon R\text{-mod} \to S\text{-mod}$ which does not send projective modules to projective. [Hint: take $R = \mathbb{Z}_2$, $S = \mathbb{Z}$.]

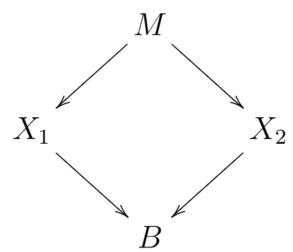
(b) Pair of functors $F\colon R\text{-mod} \to S\text{-mod}$, $G\colon S\text{-mod} \to R\text{-mod}$ is called *adjoint* if there are functorial isomorphisms $\mathrm{Hom}_R(A, G(B)) \simeq \mathrm{Hom}_S(F(A), B)$ for any R-module A and S-module B.

Prove that in such a situation, if G is exact, then F sends projective modules to projective.

3. Given a category $\mathcal{C}$, an object Z is called *universal terminal object* if for every object M there exists a unique morphism $M \to Z$.

(a) Let X_1, X_2 be two sets. Consider the category $\mathcal{C}_{X_1,X_2}$ whose objects are triples (M, p_1, p_2), where M is a set, and $p_i \colon M \to X_i$ are some maps. Morphisms in this category are defined as maps which commute with p_i (details are left to you). Prove that the universal terminal object in this category is the cartesian product $X_1 \times X_2$ (with the obvious projections $X_1 \times X_2 \to X_i$).

(b) Consider generalization of this situation, when we have three sets B, X_1, X_2 together with maps $\pi_i \colon X_i \to B$. Define the *fiber product* $X_1 \times_B X_2$ as the universal terminal object in the category $\mathcal{C}_{B;X_1,X_2}$ whose objects are triples (M, p_1, p_2), where M is a set, and $p_i \colon M \to X_i$ are maps such that the diagram



is commutative.

Give an explicit construction of $X_1 \times_B X_2$ as a certain subset in $X_1 \times X_2$.

4. An ideal I in a ring R is called *nilpotent* is there exists $n \geq 0$ such that $I^n = 0$. Prove that if R is a finite-dimensional semisimple algebra with unit over $\mathbb{C}$ then it has no nonzero nilpotent two-sided ideals.

5. (a) Let V be a complex finite-dimensional representation of a group G which is completely reducible: $V \simeq \bigoplus n_i V_i$, where V_i are irreducible and pairwise non-isomorphic. Prove that then $\mathrm{End}_G(V) \simeq \bigoplus \mathrm{Mat}_{n_i}(\mathbb{C})$, where $\mathrm{Mat}_n(\mathbb{C})$ stands for the algebra of $n \times n$ matrices over $\mathbb{C}$ and

$$\mathrm{End}_G(V) = \{f \colon V \to V \mid f \text{ commutes with the action of } G\}$$

(b) Let $V = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. Find the dimension of the space of linear operators $V \to V$ which commute with the natural action of the symmetric group S_3.

6. Let G be a group and V a finite-dimensional *real* representation of V. Define $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$; it has a natural structure of a complex representation of G.

(a) Prove that if V is irreducible, then there are two possibilities: either $V_{\mathbb{C}}$ is irreducible, or $V_{\mathbb{C}} = W_1 \oplus W_2$, where each W_i is an irreducible complex representation of G. Moreover, in the latter case, each of W_1, W_2, considered as a *real* representation of G is isomorphic to V. [Hint: consider $V - \mathbb{C}$ as a real representation of G.]

(b) Prove that $\mathrm{End}_G(V_{\mathbb{C}}) = \mathrm{End}_G(V) \otimes_{\mathbb{R}} \mathbb{C}$ and deduce from this that $V_{\mathbb{C}}$ is irreducible iff $\mathrm{End}_G(V) = \mathbb{R}$.

(c) Prove that any irreducible real representation of a commutative group is either one- or two-dimensional.