

MAT 534: HOMEWORK 8

DUE WED, NOV 14

Throughout this assignment, \mathbb{F} is an arbitrary field.

1. Which of the following rings are fields? integral domains? In each case, find all invertible elements (also called *units*)

(a) $R = \mathbb{F}[x]$

(b) $R = \mathbb{Z}[\omega]$, where $\omega \in \mathbb{C}$ is a primitive cubic root of unity.

(c) $R = \mathbb{R}[A]$ where $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

(d) $R = \mathbb{R}[A]$ where $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

(e) $R = \mathbb{Z}/n\mathbb{Z}$

2. For each of the quotient rings below, answer the following questions: is it a field? is it finite? is it isomorphic to any ring of Problem 1?

(a) $\mathbb{Z}[i]/(2)$ (b) $\mathbb{Z}[i]/(i+1)$ (c) $\mathbb{R}[x]/(x-1)^2$

(d) $\mathbb{R}[x]/(x^2+1)$ (e) $\mathbb{Z}[x]/(2, x)$ (f) $\mathbb{R}[x, y]/(xy)$

3. Let $d \in \mathbb{Z}$, $d > 1$ be squarefree (i.e., d is not divisible by a square of any prime number).

(a) Show that $\mathbb{Q}[\sqrt{d}] = \{a + b\sqrt{d}, a, b \in \mathbb{Q}\}$ is a field.

(b) Show that $\mathbb{Z}[\sqrt{d}] = \{a + b\sqrt{d}, a, b \in \mathbb{Z}\}$ is an integral domain.

(c) Define “conjugation” $\bar{} : \mathbb{Q}[\sqrt{d}] \rightarrow \mathbb{Q}[\sqrt{d}]$ by $\overline{a + b\sqrt{d}} = a - b\sqrt{d}$. Prove that then $\overline{x + y} = \bar{x} + \bar{y}$, $\overline{xy} = \bar{x} \cdot \bar{y}$.

(d) Show that $u \in \mathbb{Z}[\sqrt{d}]$ is a unit (i.e., has a multiplicative inverse in $\mathbb{Z}[\sqrt{d}]$) iff $u\bar{u} = \pm 1$.

4. Using the previous problem, show that the set of all solutions of the *Pell equation* $a^2 - db^2 = 1$, $a, b \in \mathbb{Z}$, has a structure of an abelian group. Prove that equation $a^2 - 5b^2 = 1$ has infinitely many integer solutions. (Hint: one solution is $(9, 4)$.)

5. Let $\mathbb{F}[[x]]$ be the set of all formal power series in variable x with coefficients in a field \mathbb{F} . Prove that $\mathbb{F}[[x]]$ is a ring, and that $a_0 + a_1x + a_2x^2 + \dots$ is a unit in this ring iff $a_0 \neq 0$.

6. Let \mathbb{F}_p be the finite field with p elements (p is prime). Compute

(a) the number of one-dimensional subspaces in \mathbb{F}_p^n

(b) $|GL_2(\mathbb{F}_p)|$