# MAT 534: HOMEWORK 6

DUE WED, OCT. 31

Problems marked by asterisk (*) are optional.

**1.** For a vector $v \in V$ and $f \in V^*$, denote $\langle f, v\rangle := f(v) \in \mathbb{K}$. Define, for a linear operator $L\colon V_1 \to V_2$, the *adjoint* operator $L^t\colon V_2^* \to V_1^*$ by

$$\langle L^t(f), v\rangle = \langle f, L(v)\rangle$$

(a) Prove that $(AB)^t = B^t A^t$.

(b) Without using bases, show that $\operatorname{Ker} L^t = (V_2/\operatorname{Im} L)^*$. Can you describe $\operatorname{Im} L^t$ in terms of $\operatorname{Im} L$, $\operatorname{Ker} L$?

(c) Assume that $V_1, V_2$ are finite-dimensional; choose bases $v_i \in V_1$, $w_j \in V_2$ and dual bases $v^i \in V_1^*$, $w^j \in V_2^*$. Let $A$ be the matrix of $L$ in the basis $v_i$, $w_i$, and let $B$ be the matrix of $L^t$ in the basis $v^i$, $w^j$. Prove that $B$ is the transpose of $A$: $b_{ij} = a_{ji}$.

**2.** Prove the formula for Vandermonde determinant (discussed in class). (Hint: use induction and elementary row and column transformations.)

**3.** Find all eigenvectors of the operator $A$ on $\mathbb{C}^2$ with the following matrix in the standard basis:

$$A = \begin{pmatrix} 6 & -1 \\ 16 & -2 \end{pmatrix}$$

What is its Jordan form? Give an example of a basis in $\mathbb{C}^2$ such that the matrix of $A$ in this basis is the Jordan form

**4.** Let $A$ be an operator on a finite-dimensional vector space $V$. Define the exponent $e^A$ by the following power series:

$$e^A = 1 + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots$$

(you can use without proof that this sum is absolutely convergent in the natural topology on the space $\operatorname{End}(V)$.)

(a) Let $P$ be an invertible operator on $V$. Prove that $Pe^A P^{-1} = e^{PAP^{-1}}$

(b) Prove that if $A$ and $B$ commute, then $e^{A+B} = e^A e^B$

(c) Compute the exponent of the matrix

$$\begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$$

(d) Prove that if $A$ is antisymmetric (i.e. $A + A^t = 0$), then $e^A$ is orthogonal (i.e. $AA^t = 1$).

**5.** Let $A, B\colon V \to V$ be commuting operators $AB = BA$.

(a) Show that if $V_{(\lambda)}$ is the generalized eigenspace for $A$ (that is, $V_{(\lambda)} = \operatorname{Ker}(A-\lambda)^N$ for $N \gg 0$), then $B(V_{(\lambda)}) \subset V_{(\lambda)}$.

(b) Show that if $A, B$ are both diagonalizable, then they can be diagonalized simultaneously: there is a basis in which both $A, B$ are diagonal.

*(c) Diagonalize the operator $A\colon \mathbb{C}^n \to \mathbb{C}^n$ defined by

$$A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \frac{x_n + x_2}{2} \\ \frac{x_1 + x_3}{2} \\ \vdots \\ \frac{x_{n-1} + x_1}{2} \end{bmatrix}$$

(hint: this operator commutes with the cyclic permutation of $x_i$).