

MAT 534: HOMEWORK 5

DUE WED, OCT. 24

Problems marked by asterisk (*) are optional.

1. Let $L \subset \mathbb{Z}^n$ be the subgroup generated by rows of an $n \times n$ matrix A with integer entries. Show that if $\det A = 0$, then \mathbb{Z}^n/L is infinite, and if $\det A \neq 0$, then $|\mathbb{Z}^n/L| = |\det A|$.

2. Let $V' \subset V$ be a subspace.

(a) Show that there is a canonical isomorphism

$$(V/V')^* = \{f \in V^* \mid f(w) = 0 \forall w \in V'\}$$

thus, $(V/V')^*$ is naturally a subspace (not a quotient!) of V^* .

(b) More generally, show that for any vector space W , one has $\text{Hom}(V/V', W) = \{f \in \text{Hom}(V, W) \mid f(w) = 0 \forall w \in V'\}$.

3. Let $B: V \times V \rightarrow \mathbb{K}$ be a symmetric bilinear form

(a) Define, for $v \in V$, $f_v \in V^*$ by $f_v(w) = B(v, w)$. Show that $v \mapsto f_v$ is a linear map.

(b) Assume that B non-degenerate: if $B(v, w) = 0$ for all w , then $v = 0$. Prove that then the map $V \rightarrow V^*$ constructed in the previous part is an isomorphism.

4. Let T be a linear operator on the finite-dimensional space V . Suppose there is a linear operator U on V such that $TU = I$. Prove that T is invertible, i.e. has both left and right inverse, and $U = T^{-1}$. Show that this is false when V is not finite-dimensional. (Hint: Let $T = D$ be the differentiation operator on the space of polynomials.)

5. Let U and V be subspaces of the same vector space W . Verify that $U \cap V$ and $U + V = \{u + v \mid u \in U; v \in V\}$ are also subspaces. Prove that

$$\dim(U + V) = \dim(U) + \dim(V) - \dim(U \cap V)$$

6. Let $A: V \rightarrow V$ be a linear operator on a finite-dimensional space such that $A^2 = A$. Prove that then one can write $V = V_1 \oplus V_2$ so that $A|_{V_1} = \text{id}$, $A|_{V_2} = 0$, so A is the projection operator. (Hint: take $V_1 = \text{Im } A$, $V_2 = \text{Ker } A$.)