MAT 534: HOMEWORK 7

DUE MON, OCT 22

1. Find an orthonormal eigenbasis for the operator $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ (in the standard basis of $\mathbb{R}^2$)

2. Show that if B is a non-degenerate bilinear form on a vector space V of dimension n (i.e., $\operatorname{Ker} B = \{0\}$), and W is a subspace of V of dimension k, then the image $B(W, -) \subset V^*$ has dimension k, and the orthogonal complement $W^\perp$ has dimension $n - k$.

3. Let V be a finite-dimensional Euclidean space with inner product $(\, , \,)$. Prove that then, for any symmetric operator A, the form $B_A(v, w) = (Av, w)$ is also symmetric bilinear form and that conversely, every symmetric bilinear form can be written in this form for some symmetric operator A.

4. Consider the (infinite-dimensional) vector space of complex C^∞ functions on $\mathbb{R}$ with compact support. Define the inner product in this space by

$$(f, g) = \int_{\mathbb{R}} \overline{f(x)} g(x) \, dx$$

Is the operator $\frac{d}{dx}$ Hermitian? skew-Hermitian? neither?

5. Let B be a symmetric bilinear form in a finite-dimensional real vector space V.

(a) Show that then one can write $V = V_+ \oplus V_0 \oplus V_-$, where subspaces $V_\pm, V_0$ are orthogonal with respect to B (i.e., $B(v_1, v_2) = 0$ if v_1, v_2 are from different subspaces), and restriction of B to V_+ is positive definite, to V_- negative definite, and to V_0 — zero. (Hint: choose some inner product in V, write $B(v, w) = (Av, w)$ for some symmetric operator A, and then diagonalize A.)

(b) Show that if $V = V_+ \oplus V_0 \oplus V_- = V'_+ \oplus V'_0 \oplus V'_-$ are two such decompositions, then $\dim V_+ = \dim V'_+$, $\dim V_- = \dim V'_-$, $\dim V_0 = \dim V'_0$. (Hint: prove that $V'_+ \cap (V_0 \oplus V_-) = \{0\}$.)

(c) Deduce that there exists a basis in which B is diagonal, with $+1, -1$, and 0 on the diagonal, and the number of pluses, minuses, and zeros does not depend on the choice of such a basis. (This is called the *Inertia Theorem*)