MAT 534: HOMEWORK 6

DUE MON, OCT 15

1. Let $W \subset V$ be a subspace. Show that one can find a new subspace W' such that $V = W \oplus W'$. [Warning: there is no canonical way to choose W'].

2. Construct an isomorphism $(V \oplus W)^* \simeq V^* \oplus W^*$.

3. Prove the formula for Vandermonde determinant (discussed in class). (Hint: use induction and elementary row and column transformations.)

4. Find all eigenvectors of the operator A on $\mathbb{C}^2$ with the following matrix in the standard basis:

$$A = \begin{pmatrix} 6 & -1 \\ 16 & -2 \end{pmatrix}$$

Is it diagonalizable?

5. Let A be an operator on a finite-dimensional vector space V . Define the exponent e^A by the following power series:

$$e^A = 1 + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots$$

(you can use without proof that this sum is absolutely convergent in the natural topology on the space $\mathrm{End}(V)$.)

(a) Let P be an invertible operator on V. Prove that $Pe^A P^{-1} = e^{PAP^{-1}}$

(b) Prove that if A and B commute, then $e^{A+B} = e^A e^B$

(c) Compute the exponent of the matrix

$$\begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$$

(d) Prove that if A is antisymmetric (i.e. $A + A^t = 0$), then e^A is orthogonal (i.e. $AA^t = 1$).

6. Let $L\colon V \to V$ be a linear operator.

(a) Prove that L is diagonalizable iff there exists a polynomial p without multiple roots such that $p(L) = 0$.

(b) Let L be diagonalizable and let $W \subset V$ be a subspace which is stable under L: $L(W) \subset W$. Prove that then the restriction of L to W is also diagonalizable.

7. Let $A, B\colon V \to V$ be commuting operators $AB = BA$.

(a) Show that if $V_{(\lambda)}$ is the generalized eigenspace for A (that is, $V_{(\lambda)} = \mathrm{Ker}(A - \lambda)^N$ for $N \gg 0$), then $B(V_{(\lambda)}) \subset V_{(\lambda)}$.

(b) Show that if A, B are both diagonalizable, then they can be diagonalized simultaneously: there is a basis in which both A, B are diagonal.

*(c) Diagonalize the operator $A\colon \mathbb{C}^n \to \mathbb{C}^n$ defined by

$$A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \frac{x_n+x_2}{2} \\ \frac{x_1+x_3}{2} \\ \vdots \\ \frac{x_{n-1}+x_1}{2} \end{bmatrix}$$

(hint: this operator commutes with the cyclic permutation of x_i).