MAT 534: HOMEWORK 5 (CORRECTED)

DUE MON, OCT. 8

Throughout this homework, all vector spaces are considered over the field $\mathbb{K}$.

1. Let $V' \subset V$ be a subspace.

(a) Show that there is a canonical isomorphism

$$(V/V')^* = \{f \in V^* \mid f(w) = 0 \ \forall w \in V'\}$$

thus, $(V/V')^*$ is naturally a subspace (not a quotient!) of V^*.

(b) More generally, show that for any vector space W, one has $\mathrm{Hom}(V/V', W) = \{f \in \mathrm{Hom}(V, W) \mid f(w) = 0 \ \forall w \in V'\}$.

2. Let T be a linear operator on the finite-dimensional space V. Suppose there is a linear operator U on V such that $TU = I$. Prove that T is invertible, i.e. has both left and right inverse, and $U = T^{-1}$. Show that this is false when V is not finite-dimensional. (Hint: Let $T = D$ be the differentiation operator on the space of polynomials.)

3. Let V_1 and V_2 be subspaces of the same vector space V. Verify that $V_1 \cap V_2$ and $V_1 + V_2 = \{v_1 + v_2 \mid v_1 \in V_1; v_2 \in V_2\}$ are also subspaces.

(a) Prove that

$$\dim(V_1 + V_2) = \dim(V_1) + \dim(V_2) - \dim(V_1 \cap V_2)$$

(b) Show that if V is finite-dimensional, then it is possible to choose a basis $\{v_i\}_{i \in I}$ in V and two subsets $I_1, I_2 \subset I$ such that

- $\{v_i\}_{i \in I_1}$ is a basis of V_1
- $\{v_i\}_{i \in I_2}$ is a basis of V_2
- $\{v_i\}_{i \in I_1 \cup I_2}$ is a basis of $V_1 + V_2$

*(c) (optional) Formulate and prove an analog of this for infinite-dimensional case.

4. Let $A\colon V \to V$ be a linear operator on a finite-dimensional space such that $A^2 = A$. Prove that then one can write $V = V_1 \oplus V_2$ so that $A|_{V_1} = \mathrm{id}$, $A_{V_2} = 0$, so A is the projection operator. (Hint: take $V_1 = \mathrm{Im}\, A$, $V_2 = \mathrm{Ker}\, A$.)

5. Let A, B be commuting linear operators $V \to V$ such that $A^2 = A$, $B^2 = B$. Prove that then $\mathrm{Ker}(AB) = \mathrm{Ker}(A) + \mathrm{Ker}(B)$

6. For a vector $v \in V$ and $f \in V^*$, denote $\langle f, v \rangle := f(v) \in \mathbb{K}$. Define, for a linear operator $L\colon V_1 \to V_2$, the *adjoint* operator $L^t\colon V_2^* \to V_1^*$ by

$$\langle L^t(f), v \rangle = \langle f, L(v) \rangle$$

(a) Prove that $(AB)^t = B^t A^t$.

(b) Without using bases, show that $\mathrm{Ker}\, L^t = (V_2/\mathrm{Im}\, L)^*$. Can you describe $\mathrm{Im}\, L^t$ in terms of $\mathrm{Im}\, L$, $\mathrm{Ker}\, L$?

(c) Assume that V_1, V_2 are finite-dimensional; choose bases $v_i \in V_1$, $w_j \in V_2$ and dual bases $v^i \in V_1^*$, $w^j \in V_2^*$. Let A be the matrix of L in the basis v_i, w_i, and let B be the matrix of L^t in the basis v^i, w^j. Prove that B is the transpose of A: $b_{ij} = a_{ji}$.