MAT 534: HOMEWORK 2

DUE FRI, SEPT. 14

1. Let A, B be groups and let π be an action of B on A by automorphisms: for every $b \in B$, $\pi_b \colon A \to A$ is a group automorphism. Let $G = A \times B$ (as a set) and define on it a binary operation by

$$(a,b)(a',b') = (a\pi_b(a'), bb').$$

Prove that this turns G into a group which is generated by two subgroups $\tilde{A} = \{(a, e_B)\} \simeq A$, $\tilde{B} = \{e_A, b)\} \simeq B$. Moreover, $\tilde{A}$ is normal in G and the composition morphism

$$\tilde{B} \hookrightarrow G \to G/\tilde{A}$$

is an isomorphism.

(So constructed group is called a *semidirect* product: $G = A \rtimes B$)

2. (a) Prove that the group of symmetries of an equilateral triangle is isomorphic to S_3.

(b) Prove that the group of rotations of a cube is isomorphic to S_4.[Hint: it acts on the set of diagonals...] Describe the stabilizer of a vertex; of an edge.

(c) For each pair of parallel faces of a cube consider the line passing through the centers of the faces. Using that all rotations of a cube permute these lines, construct an epimorphism $S_4 \to S_3$.

3. How many ways are there to group numbers $\{1 \dots 2n\}$ into pairs? Order of pairs and order inside each pair is not important. For example, for $n = 2$, there are three ways:

$$(12)\ (34); \qquad (13)\ (24); \qquad (14)\ (23)$$

(Hint: first show that one can define a transitive action of S_{2n} on the set of all such pairings.)

4. Let $\sigma \in S_9$ be defined by

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 8 & 5 & 9 & 7 & 4 & 1 & 2 & 6 \end{pmatrix}$$

(a) Find the cycle decomposition of σ. What is the order of σ?

(b) Find the sign of σ

5. (a) Prove that the alternating group A_n is a normal subgroup in S_n.

(b) Prove that A_n is generated by cycles of length 3.

6. (a) Describe all conjugacy classes in S_5. How many elements are in each conjugacy class?

(b) Describe all conjugacy classes in A_5. How many elements are in each conjugacy class?

7. Show that if G is a group, and $\varphi_g \colon G \to G$ is conjugation by g: $\varphi_g(x) = gxg^{-1}$, then for any automorphism σ we have $\sigma \circ \varphi_g \circ \sigma^{-1} = \varphi_{\sigma(g)}$. Deduce from this that the group $\mathrm{Inn}(G)$ of inner automorphisms is a normal subgroup in $\mathrm{Aut}(G)$.

8. Show that if $G/Z(G)$ is cyclic, then G is Abelian. (Here $Z(G)$ is the center of G.)

9. From Dummit and Foote, p. 132, problem 33