MAT 534: HOMEWORK 10

DUE WED, DEC 5

Throughout this assignment, $\mathbb{F}$ is an arbitrary field.

1. For each of the following polynomials, determine whether it is irreducible in the indicated ring. [$\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ is the finite field with p elements.]
   (a) $x^4 + 4$ in $\mathbb{Q}[x]$
   (b) $x^4 - 4x^3 + 6$ in $\mathbb{Q}[x]$
   (c) $x^2 + x + 1$ in $\mathbb{F}_2[x]$
   (d) $x^4 + 1$ in $\mathbb{F}_5[x]$
   (e) $x^4 + 10x^2 + 5x + 1$ in $\mathbb{Q}[x]$

2. Dummit and Foote, p. 312, exercise 19 (a), (b).

3. Dummit and Foote, p. 315, exercise 6. [You can use results of exercise 5 on the same page.]

4. (a) Show that if M is an R-module and N — a submodule such that $N, M/N$ are finitely generated, then M is also finitely generated.
   (b) Let M be a finitely generated module over a Noetherian ring. Prove that any submodule of M is also finitely generated. [Hint: prove this first for free modules using induction in rank.]

5. Dummit and Foote, p. 344, exercise 8.

6. Dummit and Foote, p. 344, exercise 9.

7. Let M be a module over a PID R and $a \in R$ annihilates M: $am = 0$ for any $m \in M$. Assume that $a = a_1 \dots a_n$, where a_i are pairwise relatively prime. Prove that then

$$M = M_1 \oplus \cdots \oplus M_n, \qquad M_i = \{m \in M \mid a_i M = 0\}$$

[Hint: first prove it for $n = 2$ and then use induction.]

8. Dummit and Foote, p. 356, exercise 2.