Solutions of practice midterm:

① $A \Rightarrow (B \wedge (C \vee D))$

(It can also be argued that correct interpretation is

$A \Leftrightarrow (B \wedge (C \vee D))$

② ⓐ True. Take $x=0$; then for any y, $xy=0<1$

ⓑ False. Take $x=0$; then there is no y which would give $xy=1$.

ⓒ True. Let $x>0$, $y>0$. Then $y/x>0$.

Choose any positive integer n such that $n>y/x$; then $nx>y$.

(We are using the Archimedean property of real numbers: for any positive real number t, there exists an integer $n>t$. This requires a proof; such a proof can only be given using completeness axiom of $\mathbb{R}$ and is usually done in analysis class).

③ Proof by induction:

- Base case: $n=2$.

  then $\frac{1}{3}+\frac{1}{4}=\frac{7}{12}>\frac{13}{24}$

- Induction step

  Assume $a_n > \frac{13}{24}$.

  We need to prove that then, $a_{n+1} > \frac{13}{24}$

  where $a_n = \frac{1}{n+1} + \dots + \frac{1}{2n}$.

  Indeed: $a_{n+1} = \frac{1}{n+2} + \dots + \frac{1}{2n+1} + \frac{1}{2n+2}$

$$= a_n - \frac{1}{n+1} + \frac{1}{2n+1} + \frac{1}{2n+2}$$

$$= a_n + \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{2}{2n+2}$$

$$= a_n + \frac{1}{2n+1} - \frac{1}{2n+2}$$

$$= a_n + \frac{1}{(2n+1)(2n+2)} > a_n.$$

Thus, $a_{n+1} > a_n$, so if $a_n > \frac{13}{24}$, then

$a_{n+1} > \frac{13}{24}$

(4) (a) Yes, it is an equiv. relation.

- Reflexive: $(x,y) \sim (x,y)$

  $x+y = x+y$ - true.

- Symmetric: $(x_1,y_1) \sim (x_2,y_2) \iff (x_2,y_2) \sim (x_1,y_1)$

  $(x_1+y_1 = x_2+y_2) \iff (x_2+y_2 = x_1+y_1)$

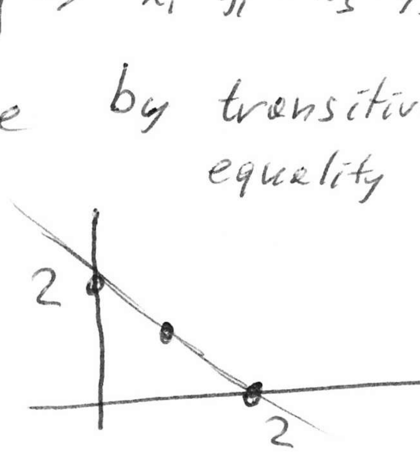
  true

- transitive: $\left.\begin{matrix}(x_1,y_1) \sim (x_2,y_2) \\ (x_2,y_2) \sim (x_3,y_3)\end{matrix}\right\} \Rightarrow (x_1,y_1) \sim (x_3,y_3)$

  $\left.\begin{matrix}x_1+y_1 = x_2+y_2 \\ x_2+y_2 = x_3+y_3\end{matrix}\right\} \Rightarrow x_1+y_1 = x_3+y_3$

  true by transitivity of equality

$[(1,1)] = \{(x,y) \mid x+y = 2\}$



(b) No. It is not transitive:

$(0,0) \sim (0,1)$

$(0,1) \sim (1,1)$

but $(0,0)$ is <u>not</u> equivalent to $(1,1)$.

⑤ Left hand side:

$$A - \bigcup B_i = \{x \mid x \in A \wedge x \notin \bigcup B_i\}$$
$$= \{x \mid x \in A \wedge (\forall i: x \notin B_i)\}$$

Thus, $x \in A - \bigcup B_i \iff x \in A \wedge (\forall i: x \notin B_i)$ (1)

RHS:

$$x \in \bigcap (A - B_i) \iff \forall i: x \in (A - B_i)$$
$$\iff \forall i: (x \in A \wedge x \notin B_i) \quad (2)$$

Thus, if $x \in A - \bigcup B_i$, then $x \in A$ and $\forall i: x \notin B_i$

then ~~$\forall i$~~: for any i, $(x \in A \wedge x \notin B_i)$ is true.

Conversely, if $x \in \bigcap A - B_i$, then

$$\forall i: (x \in A \wedge x \notin B_i)$$

so $x \in A$, and for any i, $x \notin B_i$, so

$$x \in A \wedge (\forall i: (x \notin B_i))$$

Thus; $(x \in A - \bigcup B_i) \iff (x \in \bigcap (A - B_i))$

6 (a) It is bijective: any $(y_1, y_2) \in \mathbb{R}^2$ has a unique preimage:

$$\begin{cases} x_1 - x_2 = y_1 \\ 3x_2 = y_2 \end{cases} \iff \begin{cases} x_1 = y_1 + x_2 = y_1 + \frac{1}{3} y_2 \\ x_2 = \frac{1}{3} y_2 \end{cases}$$

The inverse function is

$$f^{-1}(y_1, y_2) = \left(y_1 + \frac{1}{3} y_2, \ \frac{1}{3} y_2\right)$$

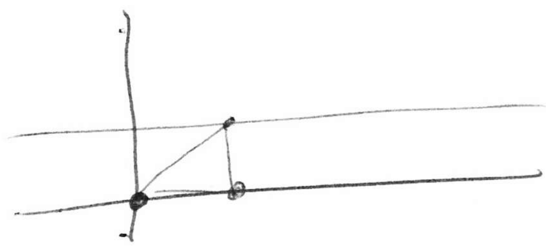
(b) ~~#~~ As in (a), it is injective, but not surjective: $(1,1)$ has no preimages (since equation $3x_2 = 1$ has no solutions with integer x_2).

(7) (a) ~~Let~~ Let $A = \mathbb{Z} \times \mathbb{Z}$ be the set of all points with integer coordinates.
~~#~~ Let T be the set of all integer triangles.
For any triangle, choose an ordering of its vertices. This gives, for any triangle t, a triple of points $(a_1, a_2, a_3) \in A \times A \times A$.
This gives injection $T \xrightarrow{\cdot\cdot} A \times A \times A$
Since cartesian product of ~~of~~ denumerable

sets is denumerable, A is denumerable, and so is $A \times A \times A$. Since any subset of a denumerable set is finite or denumerable, T is denumerable.

(b) Set of all triangles contains as subset set S of triangles ~~of the form~~ with vertices

$(0,0)$
$(0,1)$
$(1,s)$
$s \in \mathbb{R}$



Thus, S is in bijection with $\mathbb{R}$, so it is not denumerable. Thus, any set which has S as a subset is not denumerable either.