Stony Brook University
Mathematics Department
Oleg Viro

Foundations of Analysis
MAT 319/320
September 9, 2016

Homework 2, due on Thursday, September 15

1. Prove that the following statements about real numbers a, b, c are equivalent to each other:

(1) $a \in (b-c, b+c)$,
(2) $b-c < a < b+c$,
(3) $|a-b| < c$,
(4) $b \in (a-c, a+c)$,
(5) $c \in (\max(b-a, a-b), +\infty)$.

2. Let A and B be nonempty bounded subsets of $\mathbb{R}$, and let $A+B$ and $A-B$ be, respectively, the set of all sums $a+b$ and all differences $a-b$, where $a \in A$ and $b \in B$. Prove that $\sup(A+B) = \sup A + \sup B$ and $\sup(A-B) = \sup A - \inf B$.

3. Modify and prove the statements of the preceding problem for the products and quotients. Should any additional restrictions on the sets be imposed?

4. Problem 4.4 from the textbook.

Solutions

1

(1) $\Leftrightarrow$ (2): this is definition of $(b-c, b+c)$

(2) $\Leftrightarrow$ (3): if $|a-b|<c$, then

$$-c<a-b<c$$

adding b to both sides, we get

$$b-c<a<b+c$$

Conversely, if (2) holds, then subtract b to get $-c<a-b<c$

so $|a-b|<c$.

(2) $\Leftrightarrow$ (4): ~~(2) $\Leftrightarrow$ so~~ and by $|b-a|=|a-b|$, by same argument as above,

$$(3) \Leftrightarrow a-c<b<a+c$$

$$\Leftrightarrow b\in(a-c, a+c)$$

(5) $\Leftrightarrow$ (3): (5) $\Leftrightarrow$ $c>\max(a-b, b-a)$

$$\Leftrightarrow (c>a-b) \text{ and } (c>b-a)$$

$$\Leftrightarrow -c<a-b<c$$

② ⓐ For $+$:

Since for any $a \in A$, we have $a \le \sup A$
for any $b \in B$, we have $b \le \sup B$
this shows that for any $a \in A, b \in B$

$$a + b \le \sup A + \sup B$$

So $\sup A + \sup B$ is an upper bound for $A+B$; thus,

$$\sup(A+B) \le \sup(A) + \sup(B).$$

On the other hand, for any $\varepsilon > 0$, one can find $a \in A$ s.t. $a > (\sup A) - \frac{\varepsilon}{2}$ and $b \in B$ s.t. $b > (\sup B) - \frac{\varepsilon}{2}$ (otherwise, $\sup A - \frac{\varepsilon}{2}$ would be an upper bound for A). Thus,

$\forall \varepsilon > 0 \ \exists a \in A, b \in B$ s.t.

$$a + b > (\sup A + \sup B) - \varepsilon$$

Therefore, number $(\sup A + \sup B) - \varepsilon$ can not be an upper bound of $A+B$.
So, $\sup A + \sup B$ is the lowest upper bound of $A+B$.

2 (b) For $A-B$: follows from the previous and the fact that

$$\sup(-B) = -\inf(B)$$

Indeed: X is an upper bound for $-B$

$$\Updownarrow$$

$-X$ is a lower bound for B

Thus, least upper bound for $-B$

$$\|$$

$-$(greatest lower bound for B)

③ One possible modification:

if all elements of A, B are positive, then

$$\sup(A \cdot B) = \sup(A) \cdot \sup(B).$$

If, in addition, $\inf B > 0$, then

$$\sup(A/B) = \sup(A)/\inf(B)$$

Proof: If $a \in A$, then $0 < a \leq \sup(A)$

If $b \in B$, then $b \leq \sup(B)$

Thus, multiplying by a, we get

$$ab \leq a \cdot \sup(B) \leq \sup(A) \cdot \sup(B)$$

which shows that $\sup(AB) \leq \sup(A) \cdot \sup(B)$

To prove equality, it suffices to prove that for any $\varepsilon > 0$, one can find $a \in A$, $b \in B$ s.t.

$$ab > \sup(A) \cdot \sup(B) - \varepsilon$$

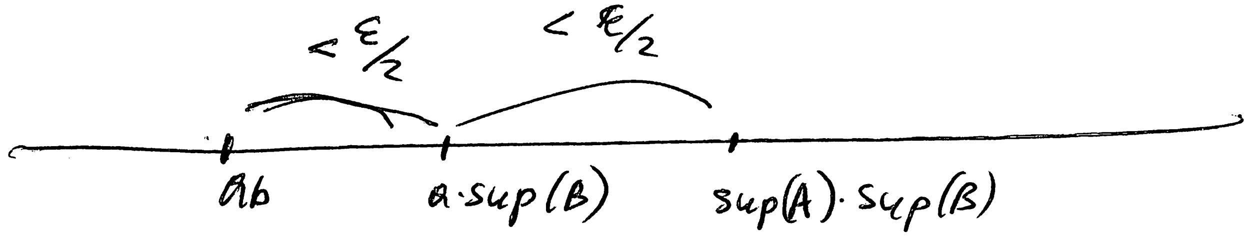
To do that, choose $a \in A$, $b \in B$ so that

$$|ab - a \cdot \sup(B)| < \varepsilon/2 \qquad (*)$$

~~$|a \cdot \sup(B)| \leq \sup(A$~~

$$|a \cdot \sup(B) - \sup(A)\sup(B)| < \varepsilon/2 \qquad (**)$$

(see figure:



To get $(*)$, take $b > \sup(B) - \frac{\varepsilon}{2a}$

To get $(**)$, take $a > \sup(A) - \frac{\varepsilon}{2 \cdot \sup(B)}$

Together $(*)$, $(**)$ imply that

$$|ab - \sup(A)\sup(B)| < \varepsilon$$

so $ab > \sup(A) \cdot \sup(B) - \varepsilon$

so $\sup(A) \cdot \sup(B) - \varepsilon$ can not be an upper bound for $A \cdot B$

Note: this was a really hard problem.

For quotients, the rule follows from the rule for products and

$$\sup(1/B) = \frac{1}{\inf(B)} \quad \text{if } \inf(B) > 0.$$

4

a) $\inf = 0$

b) $\inf = \cancel{2}\ 0$

c) $\inf = 2$

d) $\inf = e$

e) $\inf = 0$

f) $\inf = 0$

g) $\inf = 0$

h) $\inf = 2$

i) $\inf = 0$

j) $\inf = 1 - \frac{1}{3} = \frac{2}{3}$

k) $\inf = 0$

l) ~~inf~~ Not bounded below

m) $\inf = -2$

n) $\inf = -\sqrt{2}$

o) Not bounded below

p) $\inf = 1$

q) $\inf = 0$

r) $\inf = 1$

s) $\inf = 0$

t) Not bounded below

u) $\inf = 0$

v) $\inf = -1$

w) $\inf = -\frac{\sqrt{3}}{2}$