MAT 314: PRACTICE FINAL

MAY 13, 2019

The actual final will only contain 5 problems, so it will be shorter than this practice one. When solving the problems from the final exam, you are allowed to use any results discussed in class, in the textbook, or in homework problems.

Unless explicitly stated otherwise, all fileds have characteristic zero.

1. (a) Let $M = \mathbb{C}[x]/(x^2 - 5x + 6)$, considered as a module over $\mathbb{C}[x]$.
Show that then one can write $M = M_1 \oplus M_2$ (direct sum of $\mathbb{C}[x]$ -modules), where M_1, M_2 are one-dimensional as vector spaces over $\mathbb{C}$.

(b) Is it true that the $\mathbb{C}[x]$-module $M = C[x]/(x^3 - 2x^2 + x)$ can be written as direct sum of 3 modules, each of them one-dimensional over $\mathbb{C}$?

2. Let G be the abelian group generated by four generators x_1, x_2, x_3, x_4 with relations

$$3x_1 + 2x_2 - x_3 = 0$$
$$5x_1 - 2x_2 - 3x_3 + 8x_4 = 0$$
$$2x_1 + 4x_2 - 4x_4 = 0$$

Describe G as a product of cyclic groups.

3. Let G be a finite group and let V be a finite-dimensional complex representation of G. Define vector spaces V^G, V_G by

$$V^G = \{v \in V \mid gv = v \text{ for all } g \in G\}$$
$$V_G = V/V'$$

where V' is the subspace spanned by vectors $v - gv$, $v \in V$, $g \in G$.

(a) Show that $V^G \subset V, V' \subset V$ are subrepresentations.

(b) Show that for an irreducible representation V, we have $V^G = V$ if $V = \mathbb{C}$ is the trivial representation, and $V^G = 0$ otherwise. State and prove a similar result for V_G.

(c) Show that for any finite-dimensional complex representation V, we have $V^G \simeq V_G$.

4. Let $F \subset L$ be an extension and K_1, K_2 - two intermediate field extensions: $F \subset K_1 \subset L$, $F \subset K_2 \subset L$. A *composite* K_1K_2 is defined to be the smallest subfield in L containing K_1, K_2.

(a) Prove that if K_1, K_2 are finite extensions of F, then so is K_1K_2, and
$[K_1K_2 : F] \leq [K_1 : F][K_2 : F]$

(b) Prove that if K_1, K_2 are algebraic extensions of F, then so is K_1K_2.

5. Let $\alpha = \sqrt{2} + \sqrt{5} \in \mathbb{C}$.

(a) Find the degree of α over $\mathbb{Q}$ and its minimal polynomial $p(x) \in \mathbb{Q}[x]$. IS $Q(\alpha)$ nomral over $\mathbb{Q}$?

(b) Describe the Galois group G of $p(x)$ over $\mathbb{Q}$.

6. Let $x = \cos(2\pi/7)$. Prove that $\mathbb{Q} \subset \mathbb{Q}(x)$ is a normal extension; find its Galois group, and degree. Can x be written using rational numbers, arithmetic operations, and square roots? cubic roots?