

MAT 314: HOMEWORK 9
DUE TH, MAY 2, 2019

Throughout this problem set, F is a field of characteristic zero.

1. Describe the Galois group of the extension $\mathbb{Q} \subset \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$. Describe all subgroups of this Galois group and corresponding intermediate fields.
2. Let $z = e^{2\pi i/5}$. Recall that in one of the previous homeworks we have constructed a tower of extensions

$$\mathbb{Q} \subset \mathbb{Q}(t) \subset \mathbb{Q}(z), \quad t = (z + z^{-1})/2.$$

Describe the Galois group $G = \text{Gal}(\mathbb{Q}(z)/\mathbb{Q})$ and the subgroup $H = \text{Gal}(\mathbb{Q}(z)/\mathbb{Q}(t)) \subset G$.

3. For each of the following polynomials $p(x) \in \mathbb{Q}[x]$, describe its splitting field L , writing it in the form $L = \mathbb{Q}(\alpha_1, \dots, \alpha_k)$ and the Galois group $G = \text{Gal}(L/\mathbb{Q})$. For each generator of the Galois group, describe how it permutes the roots of p :
 - (a) $x^3 - 2$
 - (b) $x^4 - 2$
4. Let $K = \mathbb{Q}(\sqrt{2 + \sqrt{2}})$. Show that $\text{Gal}(K/\mathbb{Q}) = \mathbb{Z}_4$.

Hint: the roots of the corresponding irreducible polynomial are

$$\alpha_1 = \sqrt{2 + \sqrt{2}}$$

$$\alpha_2 = \sqrt{2 - \sqrt{2}}$$

$$\alpha_3 = -\alpha_1$$

$$\alpha_4 = -\alpha_2$$

Note that $\alpha_1\alpha_2 = \sqrt{2}$

5. Let $\alpha = e^{2\pi i/17}$, and let $K = \mathbb{Q}(\alpha)$.
 - (a) Show that $G = \text{Gal}(K/\mathbb{Q}) \simeq \mathbb{Z}_{16}$
 - (b) Show that it is possible to construct a sequence of subgroups $G \supset H_1 \supset H_2 \cdots \supset \{1\}$ such that $|H_i/H_{i+1}| = 2$
 - (c) Show that α is constructible (using straightedge and compass).