MAT 314: HOMEWORK 6

DUE TH, APRIL 4, 2019

Throughout this problem set, $\mathbb{F}$ is a field.

1. Let $\alpha = \sqrt{2} + \sqrt{3} \in \mathbb{R}$.
   (a) Show that $\mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{2}, \sqrt{3})$.
   (b) Find the minimal polynomial of α over $\mathbb{Q}$.
   (c) Let $\alpha' = \sqrt{2} - \sqrt{3}$. Show that there exists a field isomorphism $\mathbb{Q}(\alpha) \to \mathbb{Q}(\alpha')$ which sends α to α'.

2. For each of the following polynomials, describe its splitting field over $\mathbb{Q}$.
   (a) $x^4 + 1$
   (b) $x^3 - 5$
   (c) $x^4 + x^2 + 1$

3. Let $\mathbb{F}$ be a field of characteristic zero.
For a polynomial $f = \sum a_k x^k \in \mathbb{F}[x]$, define its derivative by

$$Df = \sum k a_k x^{k-1} \in \mathbb{F}[x].$$

   (a) Show that the derivative satisfies familiar rules:
   $D(f + g) = Df + Dg$, $D(fg) = (Df)g + f(Dg)$.
   (b) Show that if $\mathbb{E} \supset \mathbb{F}$ is an extension of $\mathbb{F}$, and $a \in \mathbb{E}$ is a root of f of order $m \geq 1$, then a is a root of Df of order $m-1$. Is this true if $\mathbb{F}$ has positive characteristic?
   (c) Show that f has no multiple roots (in any extensions of $\mathbb{F}$) iff $\gcd(f, Df) = 1$. In particular, it holds if f is irreducible.

4. Let $\mathbb{F}$ be a field of characteristic $p > 0$.
   (a) Show that the map $Fr\colon \mathbb{F} \to \mathbb{F}$ given by $Fr(x) = x^p$ is a homomorphism of fields. Deduce from this that if $\mathbb{F}$ is finite, then Fr is a bijection. [It is called the *Frobenius automorphism*].
   (b) Show that the the set $\{x \in \mathbb{F} \mid x^p = x\}$ is a subfield in $\mathbb{F}$, which is isomorphic to $\mathbb{Z}_p$. [Hint: how many different roots does the polynomial $x^p - x$ have?]