MAT 314: HOMEWORK 5

DUE TH, MARCH 14, 2019

Throughout this problem set, all representations are complex and finite-dimensional. Unless specified otherwise, G is a finite group.

1. Let V be a representation of G. Define the operator $\mathrm{Sym}\colon V \to V$ by

$$\mathrm{Sym}(v) = \frac{1}{|G|}\sum_{g\in G} \rho_g(v)$$

(a) Show that for any $v \in V$, the vector $w = \mathrm{Sym}(v)$ is invariant under action of G:

$$\rho_h(w) = w \quad \forall g \in G.$$

(b) Show that Sym is a projector: $(\mathrm{Sym})^2 = \mathrm{Sym}$.

2. Let G be a commutative (not necessarily finite) group.

(a) Prove that if V is an irreducible representation of G, then every $g \in G$ acts in V as a scalar: $\rho(g) = c \cdot I$.

(b) Prove that any irreducible representation of G is one-dimensional.

(c) Show that the previous statement would fail over $\mathbb{R}$: there exist a commutative group G which has an irreducible two-dimensional real representation.

3. (a) Describe all irreducible finite-dimensional representations of the cyclic group $G = \mathbb{Z}_n = \langle a, a^n = 1\rangle$. [Hint: use the previous problem]

(b) Consider $V = \mathbb{C}^n$ with the natural action of $\mathbb{Z}_n$ by rotations:

$$a\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_n \\ x_1 \\ \vdots \\ x_{n-1} \end{bmatrix}$$

Show that $\mathbb{C}^n$ can be written as a direct sum of irreducible representations of $\mathbb{Z}_n$. [Hint: what are eigenvalues of a in $\mathbb{C}^n$?]

4. (a) Let $U \subset \mathbb{C}^3$ be the subspace defined by

$$U = \{x \in \mathbb{C}^3 \mid x_1 + x_2 + x_3 = 0\}.$$

Prove that U is an irreducible representation of the symmetric group S_3 (where S_3 acts on $\mathbb{C}^3$ by permuting the coordinates, as described in class).

*(b) Can you prove the similar result for a subspace $U \subset \mathbb{C}^n$ and the group S_n?

5. This problem is the baby model of how group symmetry can be used to help solve various mathematical problems.

Let $A\colon \mathbb{C}^n \to \mathbb{C}^n$ be the linear operator given by

$$(Ax)_i = \tfrac{1}{2}(x_{i-1} + x_{i+1}),$$

where $i-1$, $i+1$ are taken modulo n. The goal is to diagonalize A. Straightforward approach, by writing characteristic polynomial and finding its roots, is difficult. A better way is using $\mathbb{Z}_n$ symmetry.

(a) Show that A commutes with the natural action of $\mathbb{Z}_n$ on $\mathbb{C}^n$ (see problem 3).

(b) Let

$$\mathbb{C}^n = \bigoplus V_i$$

be the decomposition of $\mathbb{C}^n$ into a direct sum of irreducible representations of $\mathbb{Z}_n$ which you found in problem 2. Use Shur's lemma to show that A preserves each of V_i's and $A|_{V_i}$ is a scalar.

(c) Find all eigenvalues and eigenvectors of A .

(d) Is it true that for any vector $x \in \mathbb{C}^n$ we have

$$\lim_{n\to\infty} A^n x = c \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \quad ?$$