# MAT 314: HOMEWORK 3

DUE TH, MARCH 1, 2019

Throughout the assignment, words PID mean "principal ideal domain". You can use all the results about PIDs discussed in class, in particular:

$$\gcd(a,b) = d \iff (a) + (b) = (d)$$
$$\operatorname{lcm}(a,b) = m \iff (a) \cap (b) = (m)$$
$$\gcd(a,b) = 1 \iff a \text{ is invertible in } R/(b)$$

**1.** Let $R$ be a PID, and $a, b \in R$ be such that $\gcd(a,b) = 1$.
  (a) Prove that $ax$ is divisible by $b$ iff $x$ is divisible by $b$.
  (b) Prove that then $\operatorname{lcm}(a,b) = ab$.

**2.** Determine the greatest common divisor in $\mathbb{Q}[x]$ of $a(x) = x^4 - 1$ and $b(x) = x^5 - 1$ and write it as a linear combination of $a(x)$ and $b(x)$.

**3.** Let $R$ be a PID.
  (a) Let $I_1 \subset I_2 \subset I_3 \cdots \subset R$ be a sequence of ideals. Prove that it stabilizes: for large enough $k$, $I_k = I_{k+1} = I_{k+2} \ldots$. [Hint: consider $I = \bigcup I_k$; then it is an ideal and thus must be generated by a single element. ]
  (b) Let $a_1, a_2, \dots \in R$ be a sequence of non-zero elements such that $a_{k+1}$ is a proper divisor of $a_k$ (i.e., $a_k/a_{k+1}$ is not invertible). Prove that this sequence can not be infinite.

**4.** Consider the ring $R = \mathbb{Z}[\sqrt{-5}] = \{a + bi\sqrt{5} \mid a, b \in \mathbb{Z}\} \subset \mathbb{C}$. For an element $z = a + ib\sqrt{5} \in R$, we denote $N(z) = z\bar{z} = a^2 + 5b^2 \in \mathbb{Z}$.
  (a) Show that $N(zw) = N(z)N(w)$.
  (b) Show that if $N(z) = 1$, then $z = \pm 1$.
  (c) Prove that there are no elements $z \in R$ with $N(z) = 2$.
  (d) Prove that elements 2, 3, $1 \pm \sqrt{-5}$ are irreducible in $R$. [Hint: if $2 = zw$, then $N(z)N(w) = N(2) = 4$.]
  (e) Show that $6 \in R$ admits two different factorizations into irreducibles in $R$. [Thus, $R$ is not a unique factorization domain and thus can not be a PID.]