MAT 314: HOMEWORK 1

DUE TH, FEB 14, 2019

Throughout this assignment, R is an arbitrary associative ring with unit (not necessarily commutative). All modules are modules over R. Letter $\mathbb{F}$ alwasy stands for a field.

1. As discussed in class, a module over the ring $\mathbb{F}[x]$ can be described as a pair (V, A), where V is a vector space over $\mathbb{F}$ and $A\colon V \to V$ is a linear operator.

(a) Give such a description for the module $M = \mathbb{F}[x]/(x^2 + 2x + 2)$, by constructing an explicit basis in the corresponding vector space V and writing the operator A as a matrix in this basis.

(b) Show that for $\mathbb{F} = \mathbb{C}$, the module M constructed in the previous part is isomorphic to a direct sum $M_1 \oplus M_2$, where M_1, M_2 are $\mathbb{C}[x]$ modules which are one-dimensional vector spaces over $\mathbb{C}$. [Hint: is A diagonalizable?]. Is the same true when $\mathbb{F} = \mathbb{R}$?

2. Let M be a an R-module and $N \subset M$, an R-submodule. Let $M' = M/N$ and denote by $f\colon M \to M'$ the obvious homomorphism of modules.

(a) Show that if $K' \subset M'$ is a submodule, then $K = f^{-1}(K') \subset M$ is also a submodule, and M/K is isomorphic to M'/K'.

(b) Show that the construction of the previous part gives a bijection

$$(\text{Submodules } K' \subset M/N) \leftrightarrow (\text{Submodules } K \subset M \text{ which contain } N)$$

3. Let M be an R-module and $M_1, M_2 \subset M$ be submodules such that $M = M_1 + M_2$, i.e. every element in M can be written as a sum $m = m_1 + m_2$, $m_1 \in M_1$, $m_2 \in M_2$ (possibly not uniquely). Prove that then one has an isomorphism $M \simeq (M_1 \oplus M_2)/(M_1 \cap M_2)$.

4. A module M over R is called *simple* if it has no nonzero proper submodules.

(a) Prove that every simple module is generated by a single element.

(b) Prove that every simple module is isomorphic to a module of the form R/I, where $I \subset R$ is a maximal left ideal.

(c) Desribe all simple modules over $\mathbb{Z}$ (i.e., abelian groups).

*(d) Describe all simple modules over $\mathbb{C}[x]$.

5. Let $R = Mat_{n\times n}(\mathbb{F})$ be the ring of $n \times n$ matrices with entries in a field F. Then $\mathbb{F}^n$ is naturally a module over R. Show that it is simple.

Hint: show that for any nonzero vector $v \in \mathbb{F}^n$, the subspace Rv contains the basis vector

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$