MAT 141 Homework 4 Solutions

1. computing the sum...

$$\sum_{i=1}^{n}(i+3)^2 = \sum_{i=4}^{n+3} i^2$$

Here we'll use a common result for the sum of squares, expressed on page 40 in your book (#6).

$$\sum_{i=4}^{n+3} i^2 = \sum_{i=1}^{n+3} i^2 - 1 - 2^2 - 3^2 = \frac{(n+3)^3}{3} + \frac{(n+3)^2}{2} + \frac{(n+3)}{6} - 14$$

which does simplify a bit but this is clear enough.

2. $\phi = \frac{1+\sqrt{5}}{2}$, $\psi = \frac{1-\sqrt{5}}{2}$

(a) Calculation. Its true.

(b) The assertion $A(n)$ is that $F(n) = \frac{\phi^n - \psi^n}{\sqrt{5}}$. We will show that $A(n)$ *and* $A(n-1)$ imply $A(n+1)$.

So assume that the assertions $A(n-1)$ and $A(n)$ hold. We can then use the formula given to calculate $F(n+1)$. That is,

$$F(n+1) = F(n-1) + F(n) = \frac{\phi^{n-1} - \psi^{n-1}}{\sqrt{5}} + \frac{\phi^n - \psi^n}{\sqrt{5}} = \frac{\phi^{n-1}(\phi+1)}{\sqrt{5}} - \frac{\psi^{n-1}(\psi+1)}{\sqrt{5}}$$

but here we substitute using what you calculated in part (a), so we get

$$F(n+1) = \frac{\phi^{n-1}(\phi+1)}{\sqrt{5}} - \frac{\psi^{n-1}(\psi+1)}{\sqrt{5}} = \frac{\phi^{n-1}(\phi^2}{\sqrt{5}} - \frac{\psi^{n-1}(\psi^2)}{\sqrt{5}} = \frac{\phi^{n+1} - \psi^{n+1}}{\sqrt{5}},$$

so $F(n+1) = \frac{\phi^{n+1} - \psi^{n+1}}{\sqrt{5}}$, which is the statement $A(n+1)$.

So we have that $A(n-1)$ and $A(n)$ imply $A(n+1)$. Since both $A(1)$ and $A(2)$ hold, we have by the principle of strong induction that the assertion holds for all $n \in \mathbb{P}$.

3. 1.7 #1

(a) A single point is itself a rectangle. If the points coordinates are (a, b) then the corresponding rectangle is explicitly $\{(x, y) | a \leq x \leq a, b \leq y \leq b\}$, which has area $(a-a) * (b-b) = 0$.

(b) Given a finite set $S = \{p_1, p_2, p_3, ..., p_n\}$ of points in the plane, we have that each point p_k individually has zero area. If $\alpha(S) \neq 0$ then we can arrive at a contradiction as follows: Set $S_k = S - \{p_1, p_2, ..., p_k\}$, or S without its first k elements. We have by the additive property that $\alpha(S_k) = \alpha(S_k + 1) + \alpha(\{p_{k+1}\})$, since $S_k = S_{k+1} \cup \{p_{k+1}\}$ and $S_{k+1} \cap \{p_{k+1}\} = \emptyset$. Since a single point set has zero area, we have that $\alpha(S_k) = \alpha(S_{k+1}$. Thus if $\alpha(S) > 0$ then $\alpha(S_1) > 1$ and similarly we can infer that $\alpha(S_{n-1}) > 0$. But $S_{n-1} = \{p_n\}$, a single point set, which we have shown must have zero area. This is a contradiction. Thus $\alpha(S) = 0$.

(c) This follows in the same manner as above. This whole process could be simplified by proving a form of *finite additivity*, which would state:

$$\alpha\left(\bigcup_{i=1}^{n} S_i\right) \leq \sum_{i=1}^{n} \alpha(S_i)$$

where all S_k are measurable sets in the plane. Such a statement could be easily proved, with a little effort.

4. 1.7 #2

   This proof really requires pictures, but the essential idea is that given a triangle ABC in the plane with designated base AB, we can drop a line l from C which is perpendicular to the base AB and intersects the line (not line segment) AB at a point D. l either splits ABC into the union of two right triangles ACD and BCD with one leg of each being l, or ABC was already a right triangle, or the right triangle ACD is entirely contained in the right triangle BCD (or vice versa). Try to imagine each case. In any case it can be shown using the resulting right triangles that the triangle ABC is measurable with area $1/2bh$.

5. 1.11 #1 I can't graph these easily. If you have a question come see me.

6. 1.11 #2. Again, no graphs. However, the partitions are listed below:

   (a) Not step.

   (b) Not step.

   (c) $P = \{-2, -1, 0, 1, 2\}$

   (d) $P = \{-2, -1, 0, 1, 2\}$

(e) $P = \{-2, -1.5, -.5, .5, 1.5, 2\}$

(f) $P = \{-2, -1.5, -1, -.5, 0, .5, 1, 1.5, 2\}$

7. 1.15 #1

(a) $(-1)(1) + (0)(1) + (1)(1) + (2)(1) = 2$

(b) $(-1)(1/2) + (0)(1) + (1)(1) + (2)(1) + (3)(1/2) = 4$

(c) Use linearity and (a) and (b). $= 6$

(d) Use linearity and (a). $= 4$

(e) $(-2)(1/2) + (-1)(1/2) + (0)(1/2) + (1)(1/2) + (2)(1/2) + (3)(1/2) + (4)(1/2) + (5)(1/2) = 6$

(f) $(0)(1) + (-1)(1) + (-2)(1) + (-3)(1) = -6$