MAT 141 Homework 3 Solutions

1. Existence of $\sqrt[3]{2}$

   Proof: Step 1: Let $S = \{s > 0 | s^3 < 2\}$ and $T = \{t > 0 | t^3 > 2\}$. First note that for all $s \in S, t \in T$, $s^3 < 2 < t^3 \Rightarrow t^3 - s^3 > 0$. But $t^3 - s^3 = (t - s)(t^2 + ts + s^2)$. Since s and t are both positive, $(t^2 + ts + s^2)$ is positive, and so $(t - s)$ must also be positive. Hence $t > s$, and therefore every element of T is greater than every element in S.

   Step 2: Now we will show that S has no maximum, and thus does not contain its supremum, and also that T has no minimum, and thus does not contain its infemum.

   So, to see that S has no maximum, assume for contradiction that $s_m \in S$ is a maximum. We will try to find a contradiction by finding an element in S that is larger than s_m. For reasons that will become clear soon, let us denote such an element by $s_m + h$. If there is some $h > 0$ so that $(s_m + h)$ is in S, then we have our contradiction.

   Now we will show that we can find such an h. So, $(s_m + h) \in S$ if and only if $(s_m + h) > 0$ and $(s_m + h)^3 < 2$, or rather

   $$s_m^3 + 3s_m^2 h + 3s_m h^2 + h^3 < 2$$

   If we insist that $0 < h < s_m$, then we have, by using $3s_m^2 h > 3s_m h^2$ and $s_m^2 h > h^3$:

   $$s_m^3 + 3s_m^2 h + 3s_m h^2 + h^3 < s_m^3 + 3s_m^2 h + 3s_m^2 h + s_m^2 h = s_m^3 + 7s_m^2 h$$

   and so

   $$(s_m + h)^3 < s_m^3 + 7s_m^2 h$$

   Thus, if we can find an $0 < h < s_m$ so that $s_m^3 + 7s_m^2 h < 2$, then we have by transitivity that $(s_m + h)^3 < 2$. But we can solve the above inequality for h. In other words,

   $$s_m^3 + 7s_m^2 h < 2 \Rightarrow h < \frac{2 - s_m^3}{7s_m^2}$$

   Now it is important to note that $\frac{2-s_m^3}{7s_m^2} > 0$, since $s_m^3 < 2$, and $s_m > 0$. If this were not so then we could not find an $h > 0$ that satisfied this inequality. As it is, we can. However, recall that to ease our simplification we had previously insisted that $h < s_m$, so let $m = min\{s_m, \frac{2-s_m^3}{7s_m^2}\}$. Any $h > 0$ that satisfies $h < m$ clearly satisfies $h < s_m$ and $h < \frac{2-s_m^3}{7s_m^2}$.

Since s_m and $\frac{2-s_m^3}{7s_m^2}$ are both positive, m must be positive. Therefore the interval $(0, m)$ is nonempty. In other words, we can find an h_0 with $0 < h_0 < m$.

But we have chosen this h so that $(s_m + h) > s_m$ and $(s_m + h)^3 < 2$. Therefore $(s_m + h) \in S$. This contradicts our assumption that s_m is the maximum os S. Therefore S cannot have a maximum. We can similarly show by contradiction that T has no minimum.

Step 3: We will use steps 1 and 2 to show that $(\sup S)^3 = 2$. Recall from step 1 that $s < t$ for all $s \in S$, $t \in T$. From theorem I.34 this means that $\sup S \leq \inf T$. Later on in this homework you will show that therefore $(\sup S)^3 \leq (\inf T)^3$. Furthermore, since $\sup S \notin S$, $(\sup S)^3 \geq 2$, and since $\inf T \notin T$, $(\inf T)^3 \leq 2$. But then, using the above conclusions, we have

$$2 \leq (\sup S)^3 \leq (\inf T)^3 \leq 2$$

and so $2 \leq (\sup S)^3 \leq 2 \Rightarrow (\sup S)^3 = 2$ or rather $\sup S = \sqrt[3]{2}$. Therefore $\sqrt[3]{2}$ exists.

2. I 4.4 #1 b,c

   (b) Our general assertion is $A(n) : 1 + 2 + 5 + \cdots + (2n - 1) = n^2$. $A(1) : 1 = 1$, which is true. Now, if we are given that $A(k)$ is true, then we have that

$$A(n) : 1 + 2 + 5 + \cdots + (2k - 1) = k^2$$

   and therefore that

$$A(n) : 1+2+5+\cdots+(2n-1)+(2(n+1)-1) = n^2+(2(n+1)-1) = n^2+2n+1$$

   which factors, so we have

$$1 + 2 + 5 + \cdots + (2n - 1) + (2(n + 1) - 1) = (n + 1)^2$$

But this is just the statement $A(k+1)$.

Therefore $A(1)$ holds and $A(k) \Rightarrow A(k+1)$, so by induction, the general assertion holds.

(c) Our general assertion is $A(n) : 1^3+2^3+\cdots+n^3 = (1+2+\cdots+n)^2$. $A(1)$ is the statement $1 = 1$, which is still true. Now we must show that $A(n) \Rightarrow A(n+1)$, so let us suppose that we are given $A(n)$. To the right side of the equation, we use the result of part (a) of this exercise. In other words:

$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2 = \left(\frac{n(n+1)}{2}\right)^2$$

. Adding $(n+1)^3$ to both sides of this equation and expanding gives:

$$1^3+2^3+\cdots+n^3+(n+1)^3 = \frac{n^2(n+1)^2}{4}+(n+1)^3 = \left(\frac{n^2}{4} + (n+1)\right)(n+1)^2$$

which factors, so we have

$$1^3+2^3+\cdots+n^3+(n+1)^3 = \left(\frac{n^2+4n+4}{4}\right)(n+1)^2 = \frac{(n+2)^2(n+1)^2}{4}$$

But we can again use (a) to substitute on the right side, this time going the other way, so we have:

$$1^3 + 2^3 + \cdots + n^3 + (n+1)^3 = (1 + 2 + \cdots + n + (n+1))^2$$

which is exactly $A(n+1)$. So we have shown that $A(n)$ implies $A(n+1)$, so by induction the general assertion holds.

3. 4.4 # 3

   Looking at the equations given, it seems the general law suggested is that $1+1/2+1/4+\cdots+1/2^n = 2-1/2^n$. In summation notation, this can be expressed $\sum_{i=0}^{n} 1/2^i = 2 - 1/2^n$. Rather than outline a proof by induction, I'll instead show you a faster way to prove this law.

   Note that (by elementary algebra) $1/2^k = 1/2^{k-1} - 1/2^k$. Using this to substitute in our original sum we have $\sum_{i=0}^{n} 1/2^{i-1} - 1/2^i$. But this sum *telescopes* to $2 - 1/2^n$, so we are done.

4. 4.7 # 11 a, b, e.

   (a) True. $0^4 = 0$, so $\sum_{n=0}^{100} n^4 = \sum_{n=1}^{100} n^4$.

   (b) False, because $\sum_{j=0}^{100} 2 = 2 * 101 = 202$, not 200.

   (e) False. Simply $100 * \sum_{k=1}^{100} k^2 > \sum_{k=1}^{100} k^3$.

5. 4.7 # 12

   Writing out the first few sums we have: $\sum_{k=1}^{1} \frac{1}{k(k+1)} = 1/2$, $\sum_{k=1}^{2} \frac{1}{k(k+1)} = 2/3$, $\sum_{k=1}^{3} \frac{1}{k(k+1)} = 3/4 \ldots$. At this point we can make the conjecture that $\sum_{k=1}^{n} \frac{1}{k(k+1)} = n/(n+1)$. This can be shown by using the substitution $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$, after which the sum telescopes to the desired result. Otherwise, it can be proved by induction as follows:

   In this case our statement $A(n)$ is $\sum_{k=1}^{n} \frac{1}{k(k+1)} = n/(n+1)$. Since this was conjectured based on the first three sums, clearly $A(1)$ is true. Now we must show that $A(n) \Rightarrow A(n+1)$. If we have $A(n)$, that is $\sum_{k=1}^{n} \frac{1}{k(k+1)} = n/(n+1)$, then add $1/(n+1)(n+2)$ to both sides, so that we have

   $$\sum_{k=1}^{n+1} \frac{1}{k(k+1)} = \frac{n(n+1)+1}{(n+1)(n+2)} = \frac{(n+1)^2}{(n+1)(n+2)} = \frac{n+1}{n+2}$$

   which is the assertion $A(n+1)$, so by induction the assertion holds for all $n \in \mathbb{P}$.

6. Given $0 < x < y$ prove by induction that for any $n \in \mathbb{P}$, $x^n < y^n$.

   The assertion $A(n)$ is that $x^n < y^n$, so clearly we have $A(1)$. Now, to show that $A(k) \Rightarrow A(k+1)$ we will start with $x^k < y^k$. Multiply both sides by x. Since $x > 0$ we have then $x^{k+1} < xy^k$. Now take the inequality $x < y$. Since $y > 0$ we have $y^k > 0$, and so we can multiply $x < y$ by y^k to get $xy^k < y^{k+1}$. But then by transitivity our two results above give us $x^{k+1} < y^{k+1}$, which is $A(k+1)$. Therefore $A(k) \Rightarrow A(k+1)$, and so by induction our assertion holds for all $n \in \mathbb{P}$.