MAT 141 Homework 2 Solutions

1. 3.12 4. Here is one proof using some basic set theory - if you don't understand please feel free to ask me (Rob) about it! From problem 2., we know that given $x \in \mathbb{R}$ there are some $m, n \in \mathbb{Z}$ with $m < x < n$. Thus x is in the interval (m,n) =

   $$(m, m+1) \cup [m+1, m+2) \cup [m+2, m+3) \cup \cdots \cup [n-1, n),$$

   and so x is $(m, m+1)$ or some half open interval $[m+k, m+k+1)$, where k is a positive integer less than n. If $x \in (m, m+1)$, then by the set builder definition of the interval $m \leq x < m+1$, and similarly if x is in some half open interval $[m+k, m+k+1)$ then (again, by definition) $m+k \leq x < m+k+1$. Since x is in one of these intervals, it must be true that for some integer i (which is either m or $m+k$), $i \leq x < i+1$.

   This shows that such an integer exists. Now we need to show that it is unique. So if there is another integer (call it j) with $j \leq x < j+1$, we will seek to show that $j = i$. Suppose for contradiction that $j \neq i$. Then either $j < i$ or $i < j$, by trichotomy. Without loss of generality*, assume for contradiction that $j < i$. Then, by properties of the integers, $j+1 \leq i$. But then $x < j+1 \leq i$, which is a contradiction. Therefore $j = i$.

   *Without loss of generality (WLOG) - this means that we are proving only one case, but the proof of the other case is the same, so we are still making a general conclusion that does not depend on the case we have chosen.

2. 3.12 6. First, we will prove that if $x < y$ AND $y - x > 1$, then there some integer n with $x < n < y$.

   So, given the above statements, let m be $[x]$, the greatest integer in x. Note that since $y - x > 1$, then $y > x+1$. But we have $x \leq m < x+1$, which implies $x \leq m < y$. If $m \neq x$, then $x < m < y$, if $m = x$, then we have that $x+1$ is an integer, and $x < x+1 < y$. Either way, we have found an integer between x and y.

   Now, to prove the more general case we will try to reduce it to a situation similar to the one above. So given only that $x < y$, we know that $y - x > 0$. From exercise 3, which we did in recitation, we know that we can choose $n \in \mathbb{P}$ with $1/n < y - x$. Since $n > 0$, multiplying both sides of this inequality by n gives us $1 < n(y-x) = ny - nx$. But then from the specific case above we can find an integer m between the real numbers nx and ny. So we have $nx < m < ny$. Multiplying this inequality by the positive number $1/n$ gives us $x < m/n < y$. But since m and n are integers, $m/b \in \mathbb{Q}$, by definition. Since x and y were arbitrary, it follows that between any two real numbers is at least one rational number, and so the rationals are *dense* in $\mathbb{R}$.

3. (a) Ok here's a really hard way to prove this first part! For any easier way see (b). Let $n \in \mathbb{P}$ be even and greater than 2. We have

$$(2-1)^n = 2^n - 2^{n-1} + 2^{n-2} - 2^{n-3} + \dots$$

but

$$-2^{n-1} + 2^{n-2} = -(2 * 2^{n-2}) + 2^{n-2} = -2^{n-2}$$

so

$$(2-1)^n = 2^n - 2^{n-2} - 2^{n-4} - 2^{n-6} - \dots - 4 - 1 < 2^n - 2 - 2 - 2 - 2 - \dots$$

where the right hand side of the above inequality has $n/2 + 1$ terms, and so equals

$$2^n - 2 * (n/2) = 2^n - n.$$

Therefore, if n is even and greater than 2,

$$2^n - n > (2-1)^n = 1 > 0,$$

and so $n < 2^n$. Since the integers are unbounded and increasing we can find an even integer m bigger than any real number x. But then $2^m > x$. It follows that the sequence $\{2^n | n \in \mathbb{P}\}$ is unbounded.

(b) Suppose $a_1, a_2, a_3, \dots$ is an increasing integer sequence. This means $a_1 < a_2 < a_3 < \cdots$. But $a_1 < a_2$ implies that $a_1 + 1 \leq a_2$, and $a_2 < a_3$ implies that $a_2 + 1 \leq a_3$ which in turn means that $a_1 + 2 \leq a_3$. Along these lines, for any n one can show that $a_1 + n - 1 \leq a_n$.

Now, given a real x with $a_1 < x$, by the unboundedness of the positive integers we can choose $m \in \mathbb{P}$ with $m > x - a_1$. Now $x < a_1 + m$. But from above, $a_1 + m \leq a_{m+1}$, and so by transitivity $x < a_{m+1}$. Thus, given any $x > a_1$, we have found some a_k which is greater than x. This means that the sequence $\{a_n | n \in \mathbb{P}\}$ is unbounded.

4. If you have any questions about the following answers, please ask me to explain them.

	bdd. above?	below?	min?	max?	inf?	sup?
(a)	no	no	no	no	no	no
(b)	no	yes	1	no	1	no
(c)	yes	yes	1/2	no	1/2	1
(d)	yes	yes	-1	1/2	-1	1/2