MAT 141 Homework 1 Solutions

1. (I.6) Proof: $a+0 = a*1+0 = a*(1+0)$, by Axioms 4 & 3. By Axiom 3, $a(1+0) = a*1 + a*0$, which by Axiom 4 is the same as $a + a*0$. So now we have shown that $a+0 = a+a*0$. By Thm. I.1 this implies that 0 = a*0. $\square$

   I.11 Proof: Given $ab = 0$, if $a \neq 0$ then by axiom 6 we can multiply both sides by a^{-1}, yielding $a * a^{-1} * b = a^{-1} * 0$. The right side of this equation is 0 by thm I.6, so $a * a^{-1} * b = b = 0$. Symmetrically, if $b \neq 0$ then $a = 0$ (this means it is the same proof substituting b for a). Therefore both a and b cannot be nonzero. Therefore if $ab = 0$ either $a = 0$, $b = 0$, or both a and b are zero. $\square$

2. 3.3 3. From the axioms, $1 * 1^{-1} = 1 = 1 * 1$. Using thm. 1.7 (the cancelation law for multiplication), we can cancel 1 from both sides, giving $1^{-1} = 1$. $\square$

   4. Assume for contradiction that there is some number y such that $0*y = 1$. But by thm I.6, $0*y = 0$, which is a contradiction. Therefore there cannot be such a number y.$\square$

3. I.22 Proof: If $a < b$, then by definition $b-a$ is positive. Similarly, $c < 0$ implies, using previous theorems, that $0 - c = 0 + -c = -c$ is positive. By axiom 7, the product of two positive numbers is itself positive, so, again using our theorems from the previous section, $-c(b-a) = -bc - (-c)a = -bc + ac = ac - bc$ is positive. But this means (by definition) that $bc < ac$, and therefore that $ac > bc$.

   I.23 $a < b$ means that $b - a$ is positive. But from previous theorems $b - a = b + -a = -a + b = (-a) - (-b)$, so $(-a) - (-b)$ is positive, which means $-b < -a$, which means $-a > -b$.$\square$

4. 2. If $x = 0$ then $x^2 + 1 = 0 * 0 + 1 = 1 \neq 0$, so assume $x \neq 0$. If $x \neq 0$, then by thm. I.20 x^2 is positive. Since 1 is positive by thm. I.21, $x^2 + 1$ is therefore positive by axiom 7, or rather $x^2 + 1 > 0$. By the trichotomy law (thm. I.16) this means that $x^2 + 1 \neq 0$.$\square$

3. Let a and b be any two negative numbers. Note that by axiom 8, $-a$ and $-b$ are positive. Therefore, by axiom 7, $-a + -b$ is positive. Now, $a+b = -(-a-b)$ by exercise 5 in section 3.3. But since $-a-b = -a+-b$ is positive, $-(-a + -b)$ is negative. Thus the sum of any two negative numbers is negative.□

4. Note that for any a, $a * (1/a) = 1 > 0$, by thm I.21. If $a > 0$, then by thm I.24, $(1/a) > 0$.□

5. The set A is closed under addition (the summation of two numbers in A is itself in A), because $a + b\sqrt{2} + c + d\sqrt{2} = (a + c) + (b + d)\sqrt{2}$. Similarly, A is closed under multiplication because $(a+b\sqrt{2})(c+d\sqrt{2}) = (ac + 2bd) + (ad + bc)\sqrt{2} \in A$. Therefore addition and multiplication are well defined over A, and the validity of the first three axioms is inherited from the real numbers. Since $0, 1 \in A$, Axiom 4 is satisfied. Axiom 5 is satisfied because for any element $a+b\sqrt{2} \in A$, $-(a+b\sqrt{2}) = -a - b\sqrt{2} \in A$. To see that Axiom 6 is satisfied, we will first show that if $a + b\sqrt{2} \in A$ and $a + b\sqrt{2} \neq 0$, then $a^2 - 2b^2 \neq 0$. If $a, b \neq 0$, $a^2 - 2b^2 = 0 \Rightarrow \sqrt{2} = a/b \Rightarrow \sqrt{2} \in \mathbb{Q}$, which is a contradiction. Therefore either one of a and b is zero, in which case either $a + b\sqrt{2}$ is zero or $a^2 - 2b^2$ is nonzero. So given $a + b\sqrt{2} \neq 0$, we have $a^2 - 2b^2 \neq 0$. But then $\frac{a-b\sqrt{2}}{a^2-2b^2}$ exists, and is the reciprocal of $a + b\sqrt{2}$.□