Practice Final Exam

MAT 118

May 15, 2009

Name: (please print)	**ID #**:
Your recitation: Rec 1, W 5:20-6:15 Rec 2, Tu 11:20-12:15 Rec 3, Th 12:50-1:45 (circle your recitation)	

Directions: There are 7 problems on 8 pages (including the cover page) in this exam. Do all of your work in this exam booklet, and cross out any work that should be ignored. **Show your reasoning and computations — not just the answer.**

You may use a calculator and up to two "cheat sheets" of paper, with formulas. No other notes or books are allowed.

This practice is similar in difficulty and length to the actual final. However, there is no guarantee that the actual final will contain exactly same type of problems: **the actual final can contain questions on any of the topics discussed in class, including topics not covered in the practice final.**

1. Consider the election with 4 candidates A, B, C, and D, with the following preference table:

Number of voters	7	4	2
1^{st} choice	A	B	D
2^{nd} choice	B	D	A
3^{rd} choice	C	C	C
4^{th} choice	D	A	B

(a) Which candidate would win a plurality election?
Solution: Number of 1st place votes: A - 7, B - 4, C - 0, D - 2. The winner is A.

(b) Which candidate, if any, is the Condorcet winner?
Solution: In pairwise comparisons:
A vs B: 9 vs 4 - A wins
A vs C: 9 vs 4 - A wins
A vs D: 7 vs 6 - A wins
Thus, A is the Condorcet winner.

(c) Which candidate would win the Borda count?
Solution: A: $4 \times 7 + 1 \times 4 + 3 \times 2 = 38$
B: $3 \times 7 + 4 \times 4 + 1 \times 2 = 39$
C: $2 \times 7 + 2 \times 4 + 2 \times 2 = 26$
D: $1 \times 7 + 3 \times 4 + 4 \times 2 = 27$
The winner is B.

(d) Suppose C decides to drop out of the race. Which candidate would win the Borda count then?
Solution: The new table is

Number of voters	7	4	2
1^{st} choice	A	B	D
2^{nd} choice	B	D	A
3^{rd} choice	D	A	B

A: $3 \times 7 + 1 \times 4 + 2 \times 2 = 29$
B: $2 \times 7 + 3 \times 4 + 1 \times 2 = 28$
D: $1 \times 7 + 2 \times 4 + 3 \times 2 = 21$
The winner is A.

2.

In a city school board, there are 3 members, each having 1 vote, and the chairman, who has 2 votes. All decisions are made by simple majority of the votes.

(a) Write the weighted voting system describing this council, in the form $[q : w_1, w_2, \dots]$.
Solution: Total number of votes is $1+1+1+2=5$; thus, simple majority would mean at least 3 votes. The weighted voting systme is $[3 : 1, 1, 1, 2]$.

(b) Find the Banzhaf power index of each player.
Solution: Winning coalitions (with critical players underlined):

$$\{\underline{P_1}, \underline{P_2}, \underline{P_3}\},$$
$$\{P_1, P_2, \underline{P_4}\}, \quad \{P_1, P_3, \underline{P_4}\}, \quad \{P_2, P_3, \underline{P_4}\},$$
$$\{\underline{P_1}, \underline{P_4}\}, \quad \{\underline{P_2}, \underline{P_4}\}, \quad \{\underline{P_3}, \underline{P_4}\},$$

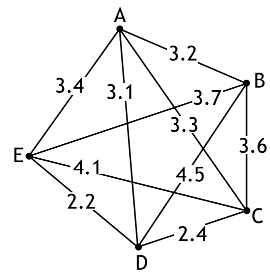
Thus, $B_1 = B_2 = B_3 = 2$, $B_4 = 6$, $T = 12$, and
$\beta_1 = \beta_2 = \beta_3 = \frac{2}{12} \approx 16.7\%$, $\beta_4 = \frac{6}{12} = 50\%$

(c) Find the Shapley-Shubik power index of each player.
Solution: There will be 24 sequential coalitions, namely:

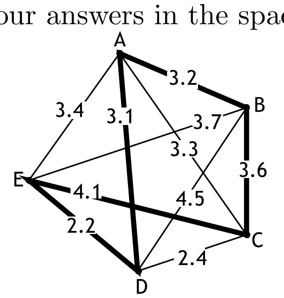
- 6 coalitions where P_4 is first:
$\{P_4, \underline{P_1}, P_2, P_3\}$, $\{P_4, \underline{P_1}, P_3, P_2\}$,
$\{P_4, \underline{P_2}, P_1, P_3\}$, $\{P_4, \underline{P_2}, P_3, P_1\}$,
$\{P_4, \underline{P_3}, P_1, P_2\}$, $\{P_4, \underline{P_3}, P_2, P_1\}$,
- 6 coalitions where P_4 is second:
$\{P_1, \underline{P_4}, P_2, P_3\}$, $\{P_1, \underline{P_4}, P_3, P_2\}$,
$\{P_2, \underline{P_4}, P_1, P_3\}$, $\{P_2, \underline{P_4}, P_3, P_1\}$,
$\{P_3, \underline{P_4}, P_1, P_2\}$, $\{P_3, \underline{P_4}, P_2, P_1\}$,
- 6 coalitions where P_4 is third:
$\{P_1, P_2, \underline{P_4}, P_3\}$, $\{P_1, P_3, \underline{P_4}, P_2\}$,
$\{P_2, P_1, \underline{P_4}, P_3\}$, $\{P_2, P_3, \underline{P_4}, P_1\}$,
$\{P_3, P_1, \underline{P_4}, P_2\}$, $\{P_3, P_2, \underline{P_4}, P_1\}$,
- 6 coalitions where P_4 is fourth:
$\{P_1, P_2, \underline{P_3}, P_4\}$, $\{P_1, P_3, \underline{P_2}, P_4\}$,
$\{P_2, P_1, \underline{P_3}, P_4\}$, $\{P_2, P_3, \underline{P_1}, P_4\}$,
$\{P_3, P_1, \underline{P_2}, P_4\}$, $\{P_3, P_2, \underline{P_1}, P_4\}$,

Thus, $s_1 = s_2 = s_3 = 4$, $s_4 = 12$, $T = 24$, so
$\sigma_1 = \sigma_2 = \sigma_3 = \frac{4}{24} \approx 16.7\%$, $\sigma_4 = \frac{12}{24} = 50\%$.
(In this case, Shapley-Shubik gives the same answer as Banzhaf; it is not always so.)

3. Consider the weighted graph shown to the right



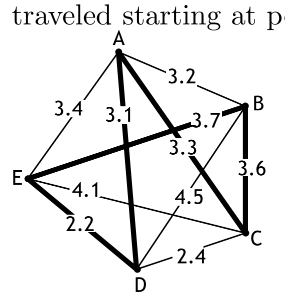
(a) Use the nearest neighbor algorithm to find an approximate solution to the traveling salesman problem, making a circuit starting at B. What is the length of this circuit? (Write your answers in the spaces below.)



Solution:

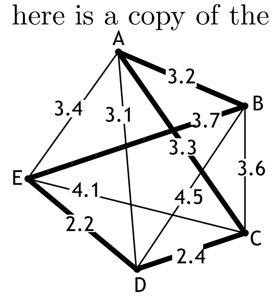
B, A, D, E, C, B. Length: $3.2 + 3.1 + 2.2 + 4.1 + 3.6 = 16.2$

(b) Use the nearest neighbor algorithm **starting at vertex A** to find an approximate solution to the traveling salesman problem, then rewrite the circuit as it would be traveled starting at point B. For your convenience, here is a copy of the graph:



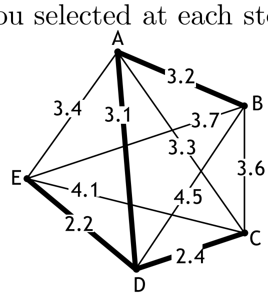
A, D, E, B, C, A.
B, C, A, D, E, B. Length: $3.1 + 2.2 + 3.7 + 3.6 + 3.3 = 15.6$

(c) Use the cheapest link algorithm to find an approximate solution to the traveling salesman problem. Write the edges you select in order, then write the circuit as it would be traveled starting at vertex B. What is the length of this circuit? For your convenience, here is a copy of the graph:



Edges selected (e.g., AB): ED, DC, AB, AC, BE
The circuit: B, E, D, C, A, B. Length: $3.7 + 2.2 + 2.4 + 3.3 + 3.2 = 14.8$

4. Consider the weighted graph below (yes, it is the same graph as in the previous problem). Find the minimal spanning tree for this graph, using Kruskal algorithm. Write the edges you selected at each step of the algorithm below.



Edges selected (e.g., AB): ED, DC, DA, AB

5. The population of a certain city is growing by 4% a year; in year 2000, the population was approxiamtely 50,000.

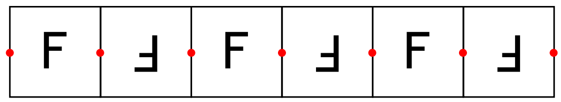
(a) Write a general formula for the population in year $2000 + n$.
Solution: Annual growth of 4% a year means that $P_{n+1} = P_n + 0.04 \cdot P_n = 1.04 \cdot P_n$. Thus, this is the exponential growth model with $r = 1.04$, so $P_n = P_0 r^n = 50,000 \cdot (1.04)^n$.

(b) By how many percents had the population grown over the period 2000 – 2010?
Solution: In 2010, the population will be $P_{10} = 50,000 \cdot (1.04)^{10} \approx 74012$. Thus, the population growth was $74012 - 50000 = 24012$; as percentage of the original population, it makes $24,012 \div 50,000 \approx 0.48 = 48\%$.

(c) In what year will the population reach 80,000?
Solution: We have already computed that in 2010 the population will not yet reach 80,000; let us compute the population for several more years:
2010: $P_{10} = 74012$
2011: $P_{11} = P_{10} \times 1.04 \approx 76972$
2012: $P_{12} = P_{11} \times 1.04 \approx 80051$
Thus, the population will reach 80,000 by year 2012.

6. For each of the border patterns shown below, describe all their symmetries. Mark on the picture rotation centers and lines of reflection symmetries (if any). Determine the symmetry type (e.g., `11`). In each case, each of the squares has side 1 cm and the pattern is continuing indefinitely in both directions — only part of it is actually shown in the figure.

(a)



Solution: Translations (specify direction and distance): yes, left and right, by multiples of 2 cm

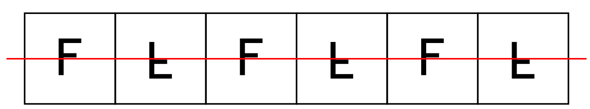
Rotations (yes/no; if yes, specify angle and mark rotocenters on the figure): yes, 180 deg, centers marked in red on the figure

Reflections (yes/no; if yes, mark reflection line on the figure): no

Glide reflections (yes/no; if yes, mark reflection line on the figure and write the translation direction and length here): no

Symmetry type:

(b)



Translations (specify direction and distance): yes, left and right, by multiples of 2 cm

Rotations (yes/no; if yes, specify angle and mark rotocenters on the figure): no

Reflections (yes/no; if yes, mark reflection line on the figure): no

Glide reflections (yes/no; if yes, mark reflection line on the figure and write the translation direction and length here): yes, reflection line marked, translation by 2 cm to the right

Symmetry type:

7. In a certain lottery, three balls are randomly drawn from a bag containing 50 balls, numbered 1 through 50.

(a) To win the lottery, you need to guess all three balls correctly (order does not matter). What is the probability of winning this lottery?
Solution: The total number of possible combinations is ${}_{50}C_3 = \frac{50\cdot 49\cdot 48}{3\cdot 2\cdot 1} = \frac{117600}{6} = 19600$. Thus, probability of guessing the answer correctly is $\frac{1}{19600} \approx 0.00005102 = 0.005\%$.

(b) What is the probability that all three drawn numbers will be even?
Solution: The answer is

$$P = \frac{\text{number of combinations of 3 even numbers}}{\text{number of all possible combinations of 3 numbers}}$$

We had computed that the total number of combinations is 19600; the number of combinations of 3 **even** numbers is ${}_{25}C_3 = \frac{25\cdot 24\cdot 23}{3\cdot 2\cdot 1} = 2300$ (because there are 25 possible even numbers in the range 1–50). Thus, the probability is

$$\frac{2300}{19600} \approx 0.117 = 11.7\%$$