THE POSITIVE MASS THEOREM FOR MULTIPLE ROTATING CHARGED BLACK HOLES

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Abstract. In this paper a lower bound for the ADM mass is given in terms of the angular momenta and charges of black holes present in axisymmetric initial data sets for the Einstein-Maxwell equations. This generalizes the mass-angular momentum-charge inequality obtained by Chrusciel and Costa to the case of multiple black holes. We also weaken the hypotheses used in the proof of this result for single black holes, and establish the associated rigidity statement.

1. Introduction

Based on heuristic arguments reminiscent of those used to motivate the Penrose inequality (see Appendix A), one may derive the following inequality

$$m^2 \geq \frac{q^2+\sqrt{q^4+4\mathcal{J}^2}}{2}, \tag{1.1}$$

relating the ADM mass m, ADM angular momentum $\mathcal{J}$, and total charge of asymptotically flat axisymmetric initial data for the Einstein-Maxwell equations. This inequality implies both the mass-angular momentum inequality $m \geq \sqrt{|\mathcal{J}|}$ and the mass-charge inequality $m \geq |q|$; the later is often referred to as the positive mass theorem with charge. While the mass-charge inequality has been rigorously established in great generality [15], without the axisymmetric assumption and for multiple black holes, the same is not true of the mass-angular momentum inequality or the mass-angular momentum-charge inequality (1.1). For these inequalities, the axisymmetric condition is necessary as it is related to conservation of angular momentum, without which the motivating heuristic arguments would no longer apply. In fact, counterexamples exist [16] without the axisymmetric hypothesis. In this setting, and with the addition of supplementary hypotheses to be discussed below, the mass-angular momentum inequality was established for a single black hole by Dain in [13], and was later extended and improved upon by Schoen and Zhou [22]. The case of multiple black holes was taken up by Chrusciel, Li, and Weinstein [9] who proved the lower bound

$$m \geq \mathcal{F}(\mathcal{J}_1, \ldots, \mathcal{J}_N), \tag{1.2}$$

where $\mathcal{F}$ is a function of the angular momentuma $\mathcal{J}_n$ associated with the N black holes. It is an open question whether this function agrees with the predicted value $\sqrt{|\mathcal{J}|}$, where $\mathcal{J} = \sum_{n=1}^{N} \mathcal{J}_n$. The inequality (1.1) has also been settled under certain conditions for single black holes by Chrusciel and Costa [7], [11]. It is the primary purpose of the present article to extend this result to the case of multiple black holes, by establishing in this setting a lower bound for the mass in the spirit of (1.2).

An initial data set (M, g, k, E, B) for the Einstein-Maxwell equations consists of a 3-manifold M, Riemannian metric g, symmetric 2-tensor k representing extrinsic curvature, and vector fields E and B which constitute the electromagnetic field. Let μ_{EM} and J_{EM} be the energy and momentum

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densities of the matter fields after contributions from the Maxwell field have been removed. If charged matter is not present, the initial data satisfy the following set of constraints

$$\begin{aligned} 16\pi\mu_{\mathrm{EM}} &= R + (\mathrm{Tr}_g\, k)^2 - |k|_g^2 - 2(|E|_g^2 + |B|_g^2), \\ 8\pi J_{\mathrm{EM}} &= \mathrm{div}_g(k - (\mathrm{Tr}_g\, k)g) + 2E\times B, \\ \mathrm{div}_g\, E &= \mathrm{div}_g\, B = 0, \end{aligned} \tag{1.3}$$

where R is the scalar curvature of g, and $(E\times B)_i = \epsilon_{ijl}E^jB^l$ is the cross product with ϵ the volume form of g.

It will be assumed throughout that the data are axially symmetric. This means that the group of isometries of the Riemannian manifold (M, g) has a subgroup isomorphic to $U(1)$, and that all quantities defining the initial data are invariant under the $U(1)$ action. Thus, if η is the Killing field associated with this symmetry, then

$$\mathfrak{L}_\eta g = \mathfrak{L}_\eta k = \mathfrak{L}_\eta E = \mathfrak{L}_\eta B = 0, \tag{1.4}$$

where $\mathfrak{L}_\eta$ denotes Lie differentiation. We will also postulate that M has at least two ends, with one designated end being asymptotically flat, and the others being either asymptotically flat or asymptotically cylindrical. Recall that a domain $M_{\mathrm{end}} \subset M$ is an asymptotically flat end if it is diffeomorphic to $\mathbb{R}^3 \setminus \mathrm{Ball}$, and in the coordinates given by the asymptotic diffeomorphism the following fall-off conditions hold

$$g_{ij} = \delta_{ij} + o_l(r^{-\frac{1}{2}}), \quad \partial g_{ij} \in L^2(M_{\mathrm{end}}), \quad k_{ij} = O_{l-1}(r^{-\lambda}), \quad \mu_{\mathrm{EM}}, J^i_{\mathrm{EM}} \in L^1(M_{\mathrm{end}}), \tag{1.5}$$

$$E^i = O_{l-1}(r^{-\lambda}), \qquad B^i = O_{l-1}(r^{-\lambda}), \qquad \lambda > \frac{3}{2}, \tag{1.6}$$

for some $l \geq 5$.[1]

Let M be simply connected. Then it is shown in [6] (see also [17] for the case when cylindrical ends are present) that $M \cong \mathbb{R}^3 \setminus \sum_{n=1}^N p_n$, and that there exists a global (cylindrical) Brill coordinate system (ρ, z, ϕ) on M, where the points p_n representing black holes all lie on the z-axis, and in which the Killing field is given by $\eta = \partial_\phi$. In these coordinates the metric takes a simple form

$$g = e^{-2U+2\alpha}(d\rho^2 + dz^2) + \rho^2 e^{-2U}(d\phi + A_\rho d\rho + A_z dz)^2, \tag{1.7}$$

where $\rho e^{-U}(d\phi + A_\rho d\rho + A_z dz)$ is the dual 1-form to $|\eta|^{-1}\eta$ and all coefficient functions are independent of ϕ. Let M^0_{end} denote the designated asymptotically flat end associated with the limit $r = \sqrt{\rho^2 + z^2} \to \infty$. Then in this end

$$U = o_{l-3}(r^{-\frac{1}{2}}), \quad \alpha = o_{l-4}(r^{-\frac{1}{2}}), \quad A_\rho = \rho o_{l-3}(r^{-\frac{5}{2}}), \quad A_z = o_{l-3}(r^{-\frac{3}{2}}). \tag{1.8}$$

The remaining ends associated with the points p_n will be denoted by M^n_{end}, and are associated with the limit $r_n \to 0$, where r_n is the Euclidean distance to p_n. As the remaining ends may be either asymptotically flat or asymptotically cylindrical, we list both types of asymptotics

$$U = 2\log r_n + o_{l-4}(r_n^{\frac{1}{2}}), \quad \alpha = o_{l-4}(r_n^{\frac{1}{2}}), \quad A_\rho = \rho o_{l-3}(r_n^{\frac{1}{2}}), \quad A_z = o_{l-3}(r_n^{\frac{3}{2}}), \tag{1.9}$$

$$U = \log r_n + O_{l-4}(1), \quad \alpha = O_{l-4}(1), \quad A_\rho = \rho o_{l-3}(r_n^{\frac{1}{2}}), \quad A_z = o_{l-3}(r_n^{\frac{3}{2}}), \tag{1.10}$$

respectively.

[1]The notation $f = o_l(r^{-a})$ asserts that $\lim_{r\to\infty} r^{a+j}\partial_r^j f = 0$ for all $j \leq l$, and $f = O_l(r^{-a})$ asserts that $r^{a+j}|\partial_r^j f| \leq C$ for all $j \leq l$. The assumption $l \geq 5$ is needed for the results in [6] and [17].

The fall-off conditions in the designated asymptotically flat end guarantee that the ADM mass, ADM angular momentum, and total charges are well-defined by the following limits

$$m = \frac{1}{16\pi}\int_{S_\infty}(g_{ij,i} - g_{ii,j})\nu^j, \tag{1.11}$$

$$\mathcal{J} = \frac{1}{8\pi}\int_{S_\infty}(k_{ij} - (\mathrm{Tr}_g\, k)g_{ij})\nu^i\eta^j + \frac{1}{4\pi}\int_{S_\infty}(E_i\nu^i)(\vec{A}_j\eta^j), \tag{1.12}$$

$$q_e = \frac{1}{4\pi}\int_{S_\infty} E_i\nu^i\,, \qquad q_b = \frac{1}{4\pi}\int_{S_\infty} B_i\nu^i\,, \tag{1.13}$$

where S_∞ indicates the limit as $r \to \infty$ of integrals over coordinate spheres S_r, with unit outer normal ν, and $\vec{A}$ is the vector potential for the magnetic field. Due to topological considerations some care must be taken to construct the vector potential, moreover its contribution to (1.12) vanishes under appropriate asymptotic conditions [14]; thus the current definition of angular momentum typically agrees the with the standard ADM notion. Here q_e and q_b denote the total electric and magnetic charge respectively, and we denote the square of the total charge by $q^2 = q_e^2 + q_b^2$. Note that the fall-off in (1.5) is not strong enough to imply that the ADM linear momentum vanishes, as is typically assumed in the study of mass-angular momentum type inequalities. Therefore the expression (1.11), which represents the ADM energy, does not necessarily coincide with the standard definition of ADM mass as the length of the 4-momentum. Nevertheless, here, we will continue to refer to (1.11) as the mass. Furthermore, note that the asymptotics (1.5) are not necessarily strong enough to guarantee that the angular momentum is finite, since the Killing fields grow like r. However, under the addition hypothesis that $J_{\mathrm{EM}}(\eta) \in L^1(M_{\mathrm{end}})$ it follows that (1.12) is finite, as may be seen from the proof of Lemma 2.1 in [14].

In the presence of an electromagnetic field, angular momentum is conserved [13], [14] if

$$J^i_{\mathrm{EM}}\eta_i = 0. \tag{1.14}$$

In this case

$$\mathcal{J} = \sum_{n=1}^{N}\mathcal{J}_n, \tag{1.15}$$

where $\mathcal{J}_n$ represents the angular momentum of the black hole p_n. Moreover, it will be shown in the next section that the condition (1.14) gives rise to a charged twist potential v which encodes the angular momentum by

$$\mathcal{J}_n = \frac{1}{4}(v|_{I_n} - v|_{I_{n-1}}), \tag{1.16}$$

where I_n denotes the interval of the z-axis between p_n and p_{n+1}, where $p_0 = -\infty$ and $p_{N+1} = \infty$. Potentials χ and ψ may also be obtained for the electric and magnetic fields, respectively, as a result of the constraints $\mathrm{div}_g\, E = \mathrm{div}_g\, B = 0$. Similarly, the charges of each black hole are given by

$$q^e_n = \frac{1}{2}(\chi|_{I_n} - \chi|_{I_{n-1}}), \qquad q^b_n = \frac{1}{2}(\psi|_{I_n} - \psi|_{I_{n-1}}), \tag{1.17}$$

with total charges

$$q^e = \sum_{n=1}^{N} q^e_n, \qquad q^b = \sum_{n=1}^{N} q^b_n. \tag{1.18}$$

In the case of a single black hole, the mass-angular momentum-charge inequality (1.1) may be established in two steps [7], [11], [22]. The first consists of proving a lower bound for the ADM mass in terms of a harmonic map energy functional

$$m \geq \mathcal{M}(U, v, \chi, \psi), \tag{1.19}$$

where

$$\mathcal{M}(U, v, \chi, \psi) = \frac{1}{8\pi}\int_{\mathbb{R}^3}\left(|\nabla U|^2 + \frac{e^{4U}}{\rho^4}|\nabla v + \chi\nabla\psi - \psi\nabla\chi|^2 + \frac{e^{2U}}{\rho^2}\left(|\nabla\chi|^2 + |\nabla\psi|^2\right)\right)dx \tag{1.20}$$

with $|\nabla U|^2 = (\partial_\rho U)^2 + (\partial_z U)^2$ and dx denoting the Euclidean volume element; this notation will be used throughout the paper. The inequality (1.19) relies heavily on the assumption of a maximal data set $\operatorname{Tr}_g k = 0$, however proposals for treating the nonmaximal case have been recently put forward in [2], [3]. The second step entails showing that the data arising from the extreme Kerr-Newman spacetime $(U_{\mathrm{KN}}, v_{\mathrm{KN}}, \chi_{\mathrm{KN}}, \psi_{\mathrm{KN}})$ (see Appendix B), minimize the functional among all data with common angular momentum and charge

$$\mathcal{M}(U, v, \chi, \psi) \geq \mathcal{M}(U_{\mathrm{KN}}, v_{\mathrm{KN}}, \chi_{\mathrm{KN}}, \psi_{\mathrm{KN}}). \tag{1.21}$$

Since the right-hand side of (1.21) agrees with the square root of the right-hand side of (1.1), together with (1.19) the desired conclusion is reached.

It should be pointed out that the hypotheses used in [7], [11], and [22] are unnecessarily strong. In these works it is assumed that the matter density is nonnegative $\mu_{\mathrm{EM}} \geq 0$, the current density vanishes $|J_{\mathrm{EM}}|_g = 0$, and that the 4-currents for the electric and magnetic fields (sources for the Maxwell equations) vanish. The later assumption concerning the 4-currents is imposed in order to secure the existence of potentials for the Maxwell field, and $|J_{\mathrm{EM}}|_g = 0$ is used to obtain a charged twist potential. Note that the use of 4-currents in general requires reference to an axisymmetric spacetime, as opposed to the initial data alone. This is justified, since in electrovacuum the existence of an axisymmetric evolution of the initial data follows from its smoothness [4], [5]. For our purposes, however, reference to the spacetime can be avoided since we will show that the potentials arise in a direct manner from the initial data, under the weakened hypotheses $\operatorname{div}_g E = \operatorname{div}_g B = J_{\mathrm{EM}}(\eta) = 0$.

Theorem 1.1. *Let (M, g, k, E, B) be a smooth, simply connected, axially symmetric, maximal initial data set satisfying $\mu_{EM} \geq 0$ and $J_{EM}(\eta) = 0$, and with two ends, one designated asymptotically flat and the other either asymptotically flat or asymptotically cylindrical. Then*

$$m^2 \geq \frac{q^2 + \sqrt{q^4 + 4\mathcal{J}^2}}{2}, \tag{1.22}$$

and equality holds if and only if (M, g, k, E, B) is isometric to the canonical slice of an extreme Kerr-Newman spacetime.

We point out that the rigidity statement of this result does not seem to have been properly established in the literature, even in the uncharged case. What has been previously established [22], is that in the case of equality the map into complex hyperbolic space arising from the given data agrees with the extreme Kerr-Newman harmonic map.

In the case of multiple black holes, the first step leading to (1.19) may be established using the same arguments as those in the single black hole case. Thus, it is in the second step (1.21) where the significant difference occurs. Here the minimizing harmonic map no longer arises from the extreme Kerr-Newman solution, or any other well known black hole solution in general. An exception happens in the special situation when all charges have the same sign and the angular momenta vanish, in which case the minimizing harmonic map arises from the Majumdar-Papapetrou solution. In the generic case, a solution $(U_0, v_0, \chi_0, \psi_0)$ to the harmonic map equations is constructed which has similar asymptotic behavior to that of the extreme Kerr-Newman map near each puncture p_n and at the designated asymptotically flat end. This asymptotic behavior allows an application of the convexity arguments in [22], showing that the constructed solution minimizes the functional $\mathcal{M}$ and yields a gap bound (see Theorem 4.1). Let

$$\mathcal{F}(\mathcal{J}_1, \dots, \mathcal{J}_N, q_1^e, \dots, q_N^e, q_1^b, \dots, q_N^b) = \mathcal{M}(U_0, v_0, \chi_0, \psi_0) \tag{1.23}$$

denote the minimum value of the functional. Our main result is as follows.

Theorem 1.2. *Let (M, g, k, E, B) be a smooth, simply connected, axially symmetric, maximal initial data set satisfying $\mu_{EM} \geq 0$ and $J_{EM}(\eta) = 0$, and with $N+1$ ends, one designated asymptotically flat and the others either asymptotically flat or asymptotically cylindrical. Then*

$$m \geq \mathcal{F}(\mathcal{J}_1, \dots, \mathcal{J}_N, q_1^e, \dots, q_N^e, q_1^b, \dots, q_N^b). \tag{1.24}$$

The functions $\mathcal{F}$ appearing in (1.2) and (1.24) agree when the charges vanish. Hence, Theorem 1.2 generalizes the result of [9] by including charge, and slightly improves this previous result in that the asymptotic assumptions on k have been weakened. Whether or not the right-hand side of (1.24) agrees with the square root of the right-hand side of (1.1) is an important open question. Note that the case of equality is not addressed in Theorem 1.2, and is closely related to the existence question for multiple rotating black hole solutions to the axisymmetric stationary electrovacuum Einstein equations. In fact we will present arguments, based on the mass gap bound of Theorem 4.1, which suggest that generically equality cannot be achieved in (1.24) when $N > 1$.

Conjecture 1.3. *Under the hypotheses of Theorem 1.2, equality in (1.24) cannot be achieved if $N > 1$ unless all charges are of the same sign and the angular momenta vanish. In this special case, the initial data set is isometric to the canonical slice of a Majumdar-Papapetrou spacetime.*

This paper is organized as follows. In the next section we describe a deformation of the Maxwell field suited for the existence of potentials, and prove Theorem 1.1. Section 3 will be devoted to the construction of a minimizer for the harmonic map functional in the case of multiple black holes, and appropriate estimates will be established. In Section 4, Theorem 1.2 will be proven and arguments supporting Conjecture 1.3 will be given. The heuristic arguments leading to (1.1) will be discussed and extended in Appendix A, to the case when several black holes are moving apart at high velocities. Lastly Appendix B is included to record several formulae associated with the Kerr-Newman and Majumdar-Papapetrou spacetimes.

2. Proof of Theorem 1.1

We first describe the construction of potentials, as alluded to in the introduction. Let

$$e_1 = e^{U-\alpha}(\partial_\rho - A_\rho \partial_\phi), \quad e_2 = e^{U-\alpha}(\partial_z - A_z \partial_\phi), \quad e_3 = \frac{1}{\sqrt{g_{\phi\phi}}}\partial_\phi, \tag{2.1}$$

be an orthonormal frame for (M, g), with dual coframe

$$\theta^1 = e^{-U+\alpha} d\rho, \quad \theta^2 = e^{-U+\alpha} dz, \quad \theta^3 = \rho e^{-U}(d\phi + A_\rho d\rho + A_z dz). \tag{2.2}$$

As in [3], consider the projections $(\overline{E}, \overline{B})$ of the electric and magnetic fields to the orbit space of η, and let F be the associated field strength defined on the auxiliary spacetime $(\mathbb{R} \times M, -dt^2 + g)$. That is

$$\overline{E}(e_i) = E(e_i), \quad \overline{B}(e_i) = B(e_i), \quad i = 1, 2, \quad \overline{E}(e_3) = \overline{B}(e_3) = 0, \tag{2.3}$$

and

$$F(e_i,\partial_t)=\overline{E}_i,\qquad F(e_i,e_j)=\epsilon_{ijl}\overline{B}^l,\qquad i,j,l=1,2,3. \tag{2.4}$$

Then

$$F(\eta,\cdot)=|\eta|\left(B(e_2)\theta^1-B(e_1)\theta^2\right),\qquad *\,F(\eta,\cdot)=|\eta|\left(E(e_2)\theta^1-E(e_1)\theta^2\right), \tag{2.5}$$

where $*$ denotes the Hodge star operation. It follows that

$$d(F(\eta,\cdot))=|\eta|(\mathrm{div}_g\, B)\theta^2\wedge\theta^1=0,\qquad d(*F(\eta,\cdot))=|\eta|(\mathrm{div}_g\, E)\theta^2\wedge\theta^1=0, \tag{2.6}$$

and hence there exist potentials for the electromagnetic field such that

$$d\psi=F(\eta,\cdot),\qquad d\chi=*F(\eta,\cdot). \tag{2.7}$$

Moreover, a calculation (Lemma 4.1 of [3]) shows that

$$d\left(k(\eta)\times\eta-\chi d\psi+\psi d\chi\right)=|\eta|\left(J_{\mathrm{EM}}(\eta)-\chi\,\mathrm{div}_g\, B+\psi\,\mathrm{div}_g\, E\right)\theta^2\wedge\theta^1=0, \tag{2.8}$$

yielding a charged twist potential satisfying

$$dv=k(\eta)\times\eta-\chi d\psi+\psi d\chi. \tag{2.9}$$

As mentioned in the introduction, the advantage of these computations is that they are made directly from the initial data, and do not require reference to the evolved spacetime. They also show clearly that the conditions $\mathrm{div}_g\, E=\mathrm{div}_g\, B=J_{\mathrm{EM}}(\eta)=0$ are necessary and sufficient for the existence of the desired potentials when M is simply connected.

It should be noted that (2.5) and (2.7) imply that χ and ψ are constant on each interval I_n of the z-axis, and

$$\begin{aligned} q^e=\frac{1}{4\pi}\int_{S_\infty}E_i\nu^i&=\sum_{n=1}^{N}\lim_{r_n\to 0}\frac{1}{4\pi}\int_{\partial B_{r_n}(p_n)}E_i\nu^i dA\\ &=\sum_{n=1}^{N}\lim_{r_n\to 0}\frac{1}{4\pi}\int_{\partial B_1(p_n)}E_i\nu^i e^{2U-\alpha}r_n^2\sin\theta d\theta d\phi\\ &=-\sum_{n=1}^{N}\lim_{r_n\to 0}\frac{1}{4\pi}\int_{\partial B_1(p_n)}|\eta|^{-1}(\partial_\theta\chi)e^{-U}r_n\sin\theta d\theta d\phi=\sum_{n=1}^{N}\frac{1}{2}(\chi|_{I_n}-\chi|_{I_{n-1}}) \end{aligned} \tag{2.10}$$

where $B_r(p_n)$ denotes the ball of radius r centered at p_n. Similar computations yield the expressions for $\mathcal{J}$ and q^b in (1.16) and (1.17).

From (2.5) and (2.7) we also find

$$|\overline{E}|_g^2+|\overline{B}|_g^2=\frac{e^{4U-2\alpha}}{\rho^2}\left(|\nabla\chi|^2+|\nabla\psi|^2\right). \tag{2.11}$$

Furthermore from (2.9) we have

$$k(e_1,e_3)=-|\eta|^{-2}\left(e_2(v)+\chi e_2(\psi)-\psi e_2(\chi)\right)=-|\eta|^{-2}e^{U-\alpha}\left(\partial_z v+\chi\partial_z\psi-\psi\partial_z\chi\right), \tag{2.12}$$

and

$$k(e_2,e_3)=|\eta|^{-2}\left(e_1(v)+\chi e_1(\psi)-\psi e_1(\chi)\right)=|\eta|^{-2}e^{U-\alpha}\left(\partial_\rho v+\chi\partial_\rho\psi-\psi\partial_\rho\chi\right). \tag{2.13}$$

It follows that

$$|k|_g^2\geq 2\left(k(e_1,e_3)^2+k(e_2,e_3)^2\right)=2\frac{e^{6U-2\alpha}}{\rho^4}|\nabla v+\chi\nabla\psi-\psi\nabla\chi|^2. \tag{2.14}$$

Recall that in Brill coordinates, the scalar curvature may be expressed simply by ([1], [12])

$$2e^{-2U+2\alpha}R = 8\Delta U - 4\Delta_{\rho,z}\alpha - 4|\nabla U|^2 - \rho^2 e^{-2\alpha}\left(A_{\rho,z} - A_{z,\rho}\right)^2, \tag{2.15}$$

where Δ is the Euclidean Laplacian on $\mathbb{R}^3$ and $\Delta_{\rho,z} = \partial_\rho^2 + \partial_z^2$. This leads to the following mass formula via an integration by parts

$$m = \frac{1}{32\pi}\int_{\mathbb{R}^3}\left(2e^{-2U+2\alpha}R + \rho^2 e^{-2\alpha}(A_{\rho,z} - A_{z,\rho})^2 + 4|\nabla U|^2\right)dx. \tag{2.16}$$

Observe that with the help of (2.11), (2.14), and the maximal data assumption, the scalar curvature may be rewritten as

$$\begin{aligned} R =& 16\pi\mu_{\mathrm{EM}} + |k|_g^2 + 2\left(|E|_g^2 + |B|_g^2\right) \\ =& 16\pi\mu_{\mathrm{EM}} + \left(|k|_g^2 - 2k(e_1,e_3)^2 - 2k(e_2,e_3)^2\right) + 2\left(E(e_3)^2 + B(e_3)^2\right) \\ &+ 2\frac{e^{6U-2\alpha}}{\rho^4}|\nabla v + \chi\nabla\psi - \psi\nabla\chi|^2 + 2\frac{e^{4U-2\alpha}}{\rho^2}\left(|\nabla\chi|^2 + |\nabla\psi|^2\right). \end{aligned} \tag{2.17}$$

Therefore

$$\begin{aligned} m - \mathcal{M}(U,v,\chi,\psi) =& \frac{1}{32\pi}\int_{\mathbb{R}^3}\left(4e^{-2U+2\alpha}\left(E(e_3)^2 + B(e_3)^2\right) + \rho^2 e^{-2\alpha}(A_{\rho,z} - A_{z,\rho})^2\right)dx \\ &+ \frac{1}{16\pi}\int_{\mathbb{R}^3} e^{-2U+2\alpha}\left(16\pi\mu_{\mathrm{EM}} + \left(|k|_g^2 - 2k(e_1,e_3)^2 - 2k(e_2,e_3)^2\right)\right)dx. \end{aligned} \tag{2.18}$$

It should be noted that this expression holds regardless of the number of ends. In the case of two ends ([7], [11], [22]), (1.21) and (2.18) yield the inequality (1.22) in Theorem 1.1.

Consider now the case of equality in (1.22). From (1.21) and (2.18), this implies that

$$\mu_{\mathrm{EM}} = 0, \qquad E(e_3) = B(e_3) = 0, \qquad A_{\rho,z} = A_{z,\rho}, \tag{2.19}$$

$$\mathcal{M}(U,v,\chi,\psi) = \mathcal{M}(U_{\mathrm{KN}}, v_{\mathrm{KN}}, \chi_{\mathrm{KN}}, \psi_{\mathrm{KN}}), \qquad k(e_i,e_j) = k(e_3,e_3) = 0, \qquad i,j \neq 3. \tag{2.20}$$

According to the gap bound in [22], a map which minimizes the functional $\mathcal{M}$ must coincide with the harmonic map associated with the extreme Kerr-Newman spacetime, that is

$$(U,v,\chi,\psi) = (U_{\mathrm{KN}}, v_{\mathrm{KN}}, \chi_{\mathrm{KN}}, \psi_{\mathrm{KN}}). \tag{2.21}$$

It follows immediately from (2.5), (2.7), and (2.19) that all components of the Maxwell field are known and agree with those induced on the $t = 0$ slice of the extreme Kerr-Newman spacetime, $(E,B) = (E_{\mathrm{KN}}, B_{\mathrm{KN}})$.

Now observe that from (2.17), (2.19), (2.20), and (2.21) we have $R = e^{-2\alpha}R_{\mathrm{KN}}$, where R_{KN} is the scalar curvature of the $t = 0$ slice of the extreme Kerr-Newman spacetime. Using the formula (2.15) produces

$$e^{-2U_{\mathrm{KN}}}R_{\mathrm{KN}} = e^{-2U+2\alpha}R = 4\Delta U_{\mathrm{KN}} - 2|\nabla U_{\mathrm{KN}}|^2 - 2\Delta_{\rho,z}\alpha. \tag{2.22}$$

However a direct computation from extreme Kerr-Newman data yields

$$e^{-2U_{\mathrm{KN}}}R_{\mathrm{KN}} = e^{-2U+2\alpha}R = 4\Delta U_{\mathrm{KN}} - 2|\nabla U_{\mathrm{KN}}|^2, \tag{2.23}$$

so that $\Delta_{\rho,z}\alpha = 0$. We claim that this, along with the asymptotics (1.8), imply that $\alpha \equiv 0$. Note that it is sufficient to show that $\alpha = 0$ along the z-axis. To see this, let $\vartheta \in (-\infty, 2\pi)$ be the cone angle deficiency [23] arising from the metric g at the axis of rotation, that is

$$\frac{2\pi}{2\pi - \vartheta} = \lim_{\rho\to 0}\frac{2\pi\cdot\mathrm{Radius}}{\mathrm{Circumference}} = \lim_{\rho\to 0}\frac{\int_0^\rho e^{-U+\alpha}d\rho}{\rho e^{-U}} = e^{\alpha(0,z)}. \tag{2.24}$$

Since (M, g) is smooth across the axis of rotation, the angle deficiency must vanish $\vartheta = 0$, and thus $\alpha(0, z) = 0$. Note that the integral in (2.24) is not exactly $2\pi \cdot$ Radius, but is rather the top order approximation to this quantity; this is all that is needed to compute the desired limit.

We are now in a position to show that (M, g) is isometric to the canonical slice of the extreme Kerr-Newman solution. By (2.19) the 1-form $A_\rho d\rho + A_z dz$ is closed, and hence there exists a potential such that $\partial_\rho f = A_\rho$ and $\partial_z f = A_z$. Consider the change of coordinates $\widetilde{\phi} = \phi + f(\rho, z)$, then the metric (1.7) takes the form

$$g = e^{-2U_{\mathrm{KN}}}(d\rho^2 + dz^2) + \rho^2 e^{-2U_{\mathrm{KN}}} d\widetilde{\phi}^2, \tag{2.25}$$

which yields the desired result $g \cong g_{\mathrm{KN}}$. Lastly, observe that (2.12), (2.13), (2.20), and $\alpha(0, z) = 0$ show that the tensor k coincides with the extrinsic curvature of the canonical extreme Kerr-Newman slice. This completes the proof of Theorem 1.1.

3. Existence of the Minimizer and Estimates

In this section we prove the existence of a minimizer for the reduced energy (1.20), having the asymptotics of extreme Kerr-Newman near each of the punctures p_n, and with prescribed angular momenta and charges. The main tool will be Theorem 2 in [25]. We denote by Γ the z-axis in $\mathbb{R}^3$ and by Γ' the axis Γ minus the N punctures p_n. The model map $\tilde{\Phi}_0 \colon \mathbb{R}^3 \setminus \Gamma \to \mathbb{H}^2_{\mathbb{C}}$ which we construct below, is not *singular Dirichlet data* as defined in Definition 2 in [25], because it does not satisfy condition (i). Nevertheless, this condition is not used in the proof of the theorem, and the only key ingredient is that the reduced energy of $\tilde{\Phi}_0$ must be finite and the pointwise tension of $\tilde{\Phi}_0$ must be bounded with appropriate decay at infinity, which will hold true for the map constructed below. Consequently, Theorem 2 in [25] can be used to conclude that there is a harmonic map $\tilde{\Psi}_0 \colon (\mathbb{R}^3 \setminus \Gamma, g_{Euc}) \to \mathbb{H}^2_{\mathbb{C}}$ which is *asymptotic* to $\tilde{\Phi}_0$, i.e. such that $\mathrm{dist}_{\mathbb{H}^2_{\mathbb{C}}}(\tilde{\Phi}_0, \tilde{\Psi}_0)$ is bounded on $\mathbb{R}^3 \setminus \Gamma$, and which satisfies the desired boundary conditions on the axis.

The *complex hyperbolic space* $\mathbb{H}^2_{\mathbb{C}}$ is the homogeneous Riemannian manifold $(\mathbb{R}^4, ds^2)$ where the metric is given by

$$ds^2 = du^2 + e^{4u}(dv + \chi d\psi - \psi d\chi)^2 + e^{2u}(d\chi^2 + d\psi^2). \tag{3.1}$$

Thus the energy density of a map $\tilde{\Phi} = (u, v, \chi, \psi)$ into $\mathbb{H}^2_{\mathbb{C}}$ is

$$\mathcal{E}(\tilde{\Phi}) = |\nabla u|^2 + e^{4u}|\nabla v + \chi \nabla \psi - \psi \nabla \chi|^2 + e^{2u}(|\nabla \chi|^2 + |\nabla \psi|^2). \tag{3.2}$$

Note that the metric (3.1) is invariant under the *translations*

$$v \mapsto v - c\psi + b\chi + a, \quad \psi \mapsto \psi + b, \quad \chi \mapsto \chi + c, \tag{3.3}$$

for any constants a, b, c. Furthermore, the action of these translations is *transitive* on any slice $\mathbb{S}_u = \{u = \text{constant}\}$, that is, $\mathbb{S}_u$ consists of a single orbit of this group action. The *tension* of the map $\tilde{\Phi}$ is the vector field $(\tau^u, \tau^v, \tau^\chi, \tau^\psi)$ on the pull-back bundle given by

(3.4)
$$\begin{aligned}
\tau^u &= \Delta u - 2e^{4u}|\nabla v + \chi \nabla \psi - \psi \nabla \chi|^2 + e^{2u}\left(|\nabla \chi|^2 + |\nabla \psi|^2\right), \\
\tau^v &= \Delta v + 2\nabla u \cdot \nabla v + 2(\nabla v + \chi \nabla \psi - \psi \nabla \chi) \cdot \nabla u - 2e^{2u}(\nabla v + \chi \nabla \psi - \psi \nabla \chi) \cdot (\chi \nabla \chi + \psi \nabla \psi), \\
\tau^\chi &= \Delta \chi + 2\nabla u \cdot \nabla \chi - 2e^{2u}(\nabla v + \chi \nabla \psi - \psi \nabla \chi) \cdot \nabla \psi, \\
\tau^\psi &= \Delta \psi + 2\nabla u \cdot \nabla \psi + 2e^{2u}(\nabla v + \chi \nabla \psi - \psi \nabla \chi) \cdot \nabla \chi.
\end{aligned}$$

Thus the tension of a map vanishes if and only if the map is harmonic.

Note that for each set of values $(\mathcal{J}_n, q_n^e, q_n^b)$, $n = 1, \ldots, N$, there is an extreme Kerr-Newman solution with this angular momentum, electric and magnetic charge, and there is a corresponding harmonic map $\tilde{\Phi}_n \colon \mathbb{R}^3 \setminus \Gamma \to \mathbb{H}^2_{\mathbb{C}}$. In view of the transitive isometric action of the translations above, this map $\tilde{\Phi}_n = (u_n, v_n, \chi_n, \psi_n)$ is only determined up to constants (a_n, b_n, c_n), where $v_n \mapsto v_n - c_n\psi_n + b_n\chi_n + a_n$, $\chi_n \mapsto \chi_n + b_n$, and $\psi_n \mapsto \psi_n + c_n$, as well as a domain translation $z \mapsto z + d$. Thus, we can set the constant values of (v_n, χ_n, ψ_n) on the component of Γ' on one side of the puncture p_n, and the values on the other side of p_n will be determined by the angular momentum, electric and magnetic charge of $\tilde{\Phi}_n$. Using this freedom we obtain $N+1$ harmonic maps $\tilde{\Phi}_n$, where for each $n = 1, \ldots N$ the values of (v_n, χ_n, ψ_n) agree with the values of (v, χ, ψ) in the map $\tilde{\Psi}$ corresponding to our given initial data set on the components of Γ' lying to both sides of the puncture p_n, and where the values of $(v_{N+1}, \chi_{N+1}, \psi_{N+1})$ agree with those of (v, χ, ψ) on the unbounded components $\Gamma'_\pm$ of the axis Γ'. Furthermore, we still have an overall translation available, i.e. a translation in $\mathbb{H}^2_{\mathbb{C}}$ which can be applied to $\tilde{\Psi}$. This will be used in the proof of Lemma 3.1 below.

Let us now construct the model map $\tilde{\Phi}_0$. For the sake of simplicity, we illustrate the construction when $N = 2$ with the help of Figure 1; clearly the construction can be carried out with any value of N. First, define the map $\tilde{\Phi}_0$ to be equal to $\tilde{\Phi}_3$ on $\mathcal{B}$, the region outside a large ball which does not intersect a neighborhood of the punctures, and to be equal to $\tilde{\Phi}_n$ on small balls $\mathcal{B}_n$ surrounding the punctures, $n = 1, 2$. These are the dark shaded regions in Figure 1. Next, define the map $\tilde{\Phi}_0$ in narrow cylindrical tubes surrounding the components of Γ' joining the different $\mathcal{B}_n$'s, the lightly shaded regions in Figure 1. Consider for example $\mathcal{C}_2$. We pick a smooth function $0 \leq \lambda(z) \leq 1$ which is 0 near $\mathcal{B}_1$ and 1 near $\mathcal{B}_2$, and define

$$\tilde{\Phi}_0 = (1-\lambda)\tilde{\Phi}_1 + \lambda\tilde{\Phi}_2 = \big((1-\lambda)u_1 + \lambda u_2, (1-\lambda)v_1 + \lambda v_2, (1-\lambda)\chi_1 + \lambda\chi_2, (1-\lambda)\psi_1 + \lambda\psi_2\big). \tag{3.5}$$

Finally, we extend $\tilde{\Phi}_0$ to the remaining region Ω, the white region in Figure 1, so that it is smooth.

Lemma 3.1. *The reduced energy of $\tilde{\Phi}_0 = (\tilde{u}_0, \tilde{v}_0, \tilde{\chi}_0, \tilde{\psi}_0)$ is finite, the tension $\tau(\tilde{\Phi}_0)$ has support inside a bounded set, and $\tau(\tilde{\Phi}_0)$ is pointwise bounded. Moreover, the values of $(\tilde{v}_0, \tilde{\chi}_0, \tilde{\psi}_0)$ agree with those of the given data (v, χ, ψ) on each component of Γ'.*

Proof. The reduced energy of the extreme Kerr-Newman harmonic map is finite, see for example [7, 11, 22]. Thus the integral of the reduced energy density over the regions $\mathcal{B}_n$ is clearly finite. Also the integral over Ω is finite. Thus, it is only necessary to check the integral over $\mathcal{C}_n$. For clarity we set $n = 1$. Since all the quantities we seek to estimate are geometric invariants, we can now use the last translation available to set all the constants $v_1|_{I_1} = v_2|_{I_1}$, $\chi_1|_{I_1} = \chi_2|_{I_1}$, $\psi_1|_{I_1} = \psi_2|_{I_1}$ to zero, where I_1 is the portion of the axis between p_1 and p_2. This implies that $\rho^{-2}|v_n|$, $\rho^{-1}|\psi_n|$ and $\rho^{-1}|\chi_n|$, $n = 1, 2$, are bounded in $\mathcal{C}_1$. We write $\tilde{U}_0 = \tilde{u}_0 + \log\rho$ and $U_n = u_n + \log\rho$, $n = 1, 2$. It follows that $\tilde{U}_0 = (1-\lambda)U_1 + \lambda U_2$. Thus, in $\mathcal{C}_1$, we have for the reduced energy density

$$\begin{aligned}\mathcal{E}'(\tilde{\Phi}_0) =& |\nabla\tilde{U}_0|^2 + \frac{e^{4\tilde{U}_0}}{\rho^4}|\nabla\tilde{v}_0 + \tilde{\chi}_0\nabla\tilde{\psi}_0 - \tilde{\psi}_0\nabla\tilde{\chi}_0|^2 + \frac{e^{2\tilde{U}_0}}{\rho^2}\left(|\nabla\tilde{\chi}_0|^2 + |\nabla\tilde{\psi}_0|^2\right)\\ \leq& |\nabla U_1|^2 + |\nabla U_2|^2 + C\left(\rho^{-4}|\nabla v_1|^2 + \rho^{-2}|\nabla\chi_1|^2 + \rho^{-2}|\nabla\psi_1|^2 + \cdots + |\nabla\lambda|^2\right),\end{aligned} \tag{3.6}$$

where C is a constant which depends on the supremum of $|U_n|$, $\rho^{-2}|v_n|$, $\rho^{-1}|\chi_n|$, $\rho^{-1}|\psi_n|$. Thus $\mathcal{E}'(\tilde{\Phi}_0)$ is bounded over $\mathcal{C}_1$, and integrating over $\mathcal{C}_1$ clearly gives a finite quantity. It follows that the integral of the reduced energy over of $\mathcal{C}_n$, $n = 2, 3$ is also finite, and hence the reduced energy of $\tilde{\Phi}_0$ is finite.

Next, consider the second claim of the lemma. Since the $\tilde{\Phi}_n$ are harmonic, $\tau(\tilde{\Phi}_n) = 0$ on $\mathcal{B}_n$, $n = 1, 2$, and $\tau(\tilde{\Phi}_3) = 0$ on $\mathcal{B}$. Therefore the support of $\tau(\tilde{\Phi}_0)$ is contained in $\Omega \cup_{n=1}^3 \mathcal{C}_n$.

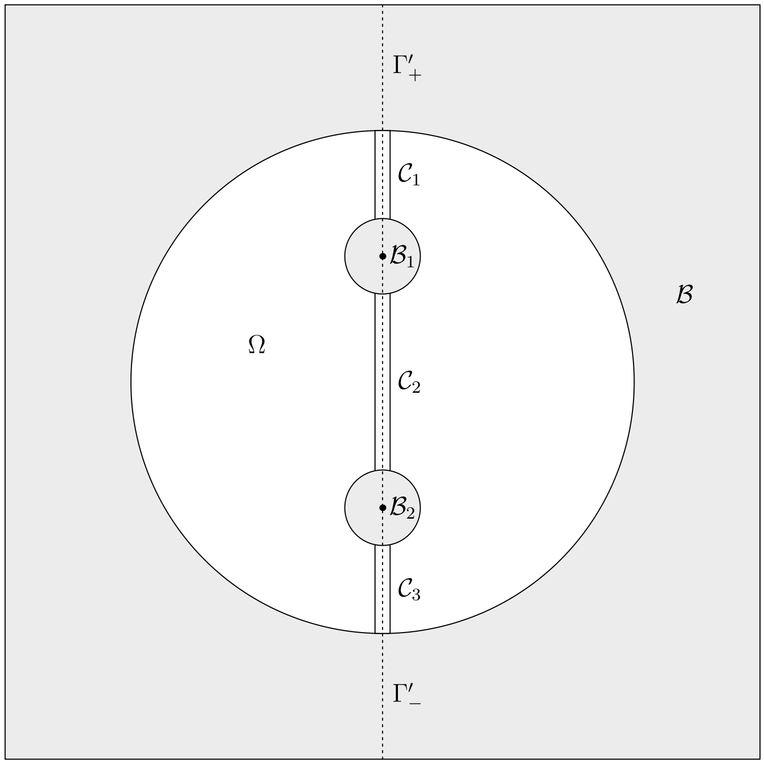

FIGURE 1. Construction of $\tilde{\Phi}_0$

For the proof of the third claim of the lemma, note that since the tension vanishes on $\mathcal{B}_n$ and is clearly bounded on Ω, it remains to check the boundedness on $\mathcal{C}_n$. Once again, we focus on $\mathcal{C}_1$. Since $\tilde{\Phi}_n$ is harmonic, we have $|\Delta U_n| \leq 2\mathcal{E}'(\tilde{\Phi}_n)$, $n = 1, 2$. Moreover, on $\mathcal{C}_1$

$$\Delta\tilde{u}_0 = (1-\lambda)\Delta U_1 + \lambda\Delta U_2 + 2\nabla\lambda\cdot\nabla(U_2 - U_1) + (U_2 - U_1)\Delta\lambda, \tag{3.7}$$

so that

$$\begin{aligned} |\tau^{\tilde{u}_0}| \leq& |\Delta\tilde{u}_0| + 2\mathcal{E}'(\tilde{\Phi}_0) \\ \leq& |\Delta U_1| + |\Delta U_2| + 2|\nabla\lambda|\left(\mathcal{E}'(\tilde{\Phi}_1) + \mathcal{E}'(\tilde{\Phi}_2)\right) + C|\Delta\lambda| + 2\mathcal{E}'(\tilde{\Phi}_0) \\ \leq& 2(1+|\nabla\lambda|)\left(\mathcal{E}'(\tilde{\Phi}_1) + \mathcal{E}'(\tilde{\Phi}_2)\right) + C|\Delta\lambda| + 2\mathcal{E}'(\tilde{\Phi}_0), \end{aligned} \tag{3.8}$$

where $C = \sup(|U_1| + |U_2|)$. It follows that $|\tau^{\tilde{u}_0}|$ is bounded. Next observe that $e^{2\tilde{u}_0}|\nabla\tilde{v}_0|$, $e^{\tilde{u}_0}|\tilde{\chi}_0|$, $e^{\tilde{u}_0}|\nabla\tilde{\chi}_0|$, $e^{\tilde{u}_0}|\tilde{\psi}_0|$ and $e^{\tilde{u}_0}|\nabla\tilde{\psi}_0|$ are all bounded on $\mathcal{C}_1$, since the same is true of $e^{2u_n}|\nabla v_n|$, $e^{u_n}|\chi_n|$, $e^{u_n}|\nabla\chi_n|$, $e^{u_n}|\psi_n|$ and $e^{u_n}|\nabla\psi_n|$, $n = 1, 2$. Using this, and $e^{2u}|\Delta v_n| \leq C\mathcal{E}'(\Phi_n)$, the proof for the $\tilde{v}_0$ component now proceeds in much the same way as for the $\tilde{u}_0$ component:

$$\begin{aligned} e^{2\tilde{u}_0}|\tau^{\tilde{v}_0}| \leq& e^{2\tilde{u}_0}|\Delta\tilde{v}_0| + C\mathcal{E}'(\tilde{\Phi}_0) \\ \leq& e^{2\tilde{u}_0}|\Delta v_1| + e^{2\tilde{u}_0}|\Delta v_2| + 2|\nabla\lambda|\left(\mathcal{E}'(\tilde{\Phi}_1) + \mathcal{E}'(\tilde{\Phi}_2)\right) + C|\Delta\lambda| + C\mathcal{E}'(\tilde{\Phi}_0) \\ \leq& (C + 2|\nabla\lambda|)\left(\mathcal{E}'(\tilde{\Phi}_1) + \mathcal{E}'(\tilde{\Phi}_2)\right) + C|\Delta\lambda| + C\mathcal{E}'(\tilde{\Phi}_0). \end{aligned} \tag{3.9}$$

It follows that $e^{2\tilde{u}_0}|\tau^{\tilde{v}_0}|$ is bounded on $\mathcal{C}_1$. In a similar way, we obtain that $e^{\tilde{u}_0}|\tau^{\tilde{\psi}_0}|$ and $e^{\tilde{u}_0}|\tau^{\tilde{\chi}_0}|$ are bounded on $\mathcal{C}_1$, and it follows that

$$\tau(\tilde{\Phi}_0) = (\tau^{\tilde{u}_0})^2 + e^{4\tilde{u}_0}(\tau^{\tilde{v}_0} + \tilde{\chi}_0\tau^{\tilde{\psi}_0} - \tilde{\psi}_0\tau^{\tilde{\chi}_0})^2 + e^{2\tilde{u}_0}\left((\tau^{\tilde{\chi}_0})^2 + (\tau^{\tilde{\psi}_0})^2\right) \tag{3.10}$$

is bounded on $\mathcal{C}_1$.

Lastly, it is immediately apparent from the construction that the values of the potentials for the model map agree with those of the given data on the axis. □

Corollary 3.2. *For any set of punctures p_n on the axis Γ and prescribed constants $v_0|_{I_n}$, $\chi_0|_{I_n}$, $\psi_0|_{I_n}$, $n = 1, \ldots, N$, there exists a corresponding unique harmonic map $\tilde{\Psi}_0 = (u_0, v_0, \chi_0, \psi_0)$ which is asymptotic to $\tilde{\Phi}_0$, and satisfies*

$$U_0 = u_0 + \log\rho = \log r_n + O(1), \qquad v_0, \chi_0, \psi_0 = O(1), \quad \textit{as} \quad r_n \to 0. \tag{3.11}$$

Proof. The existence of $\tilde{\Psi}_0$ and the fact that it is asymptotic to $\tilde{\Phi}_0$, follow from the main result in [25] combined with Lemma 3.1. In order to establish uniqueness, assume that there are two solutions $\tilde{\Psi}_0$ and $\tilde{\Psi}_1$. Since the target space has negative curvature, the distance function $f(x) = \mathrm{dist}_{\mathbb{H}^2_{\mathbb{C}}}(\tilde{\Psi}_0(x), \tilde{\Psi}_1(x)) \geq 0$ is subharmonic on $\mathbb{R}^3 \setminus \Gamma$. From this, it follows by a maximum principle type argument (Proposition C.4 in [9]) that $f \equiv 0$. □

As a consequence of the fact that $\tilde{\Psi}_0$ is asymptotic to the model map $\tilde{\Phi}_0$ at spatial infinity, that is $\mathrm{dist}_{\mathbb{H}^2_{\mathbb{C}}}(\tilde{\Phi}_0, \tilde{\Psi}_0) \to 0$ as $r \to \infty$, we obtain

$$|U_0| \leq C \quad \text{on} \quad \mathbb{R}^3 \setminus \cup_{n=1}^N B_\delta(p_n), \qquad |v_0| + |\chi_0| + |\psi_0| \leq C \quad \text{on} \quad \mathbb{R}^3. \tag{3.12}$$

It is expected [9] that more refined asymptotic fall-off estimates should hold for $\tilde{\Psi}_0$, which are in line with (4.8)-(4.19).

In order to show that the solution Ψ_0 minimizes the functional (1.20), we will need certain estimates concerning the asymptotics.

Proposition 3.3. *On $\mathbb{R}^3 \setminus \Gamma$, the solution $\Psi_0 = (U_0, v_0, \chi_0, \psi_0)$ satisfies the estimate*

$$|\nabla(U_0 - \log\rho)|^2 + \frac{e^{4U_0}}{\rho^4}|\nabla v_0 + \chi_0\nabla\psi_0 - \psi_0\nabla\chi_0|^2 + \frac{e^{2U_0}}{\rho^2}(|\nabla\chi_0|^2 + |\nabla\psi_0|^2) \leq \frac{C}{\rho^2}, \tag{3.13}$$

and in particular near each puncture p_n it holds that

$$|\nabla U_0| \leq \frac{C}{\rho}, \qquad |\nabla v_0 + \chi_0\nabla\psi_0 - \psi_0\nabla\chi_0| \leq \frac{C\rho}{r_n^2}, \qquad |\nabla\chi_0| + |\nabla\psi_0| \leq \frac{C}{r_n}. \tag{3.14}$$

Proof. This result is analogous to (2.20) in [9], and the proof follows in the same way. As details of the proof were left out in [9], we provide them here in the current and more general setting.

Let x be a point of Euclidean distance ρ from the axis Γ, and let $B_{\rho/2}(x)$ be a Euclidean ball of radius $\rho/2$. By the Bochner identity (equation (12) in [25]), and the fact that the target manifold $\mathbb{H}^2_{\mathbb{C}}$ has negative curvature, it follows that the energy density $\mathcal{E}(\tilde{\Psi}_0)$ is subharmonic $\Delta\mathcal{E}(\tilde{\Psi}_0) \geq 0$. Thus, from the mean value theorem we obtain

$$\mathcal{E}(\tilde{\Psi}_0)|_x \leq \fint_{B_{\rho/2}(x)} \mathcal{E}(\tilde{\Psi}_0), \tag{3.15}$$

where $\fint_{B_{\rho/2}} = \mathrm{vol}(B_{\rho/2})^{-1}\int_{B_{\rho/2}}$ denotes the average. Now observe that $\mathcal{E}(\tilde{\Psi}_0) \leq \mathcal{E}'(\Psi_0) + C/\rho^2$, and taking the average of this inequality over $B_{\rho/2}(x)$ produces

$$\mathcal{E}(\tilde{\Psi}_0)|_x \leq \fint_{B_{\rho/2}(x)} \mathcal{E}'(\Psi_0) + \frac{C}{\rho^2}. \tag{3.16}$$

Similarly we have $\mathcal{E}'(\Psi_0) \leq \mathcal{E}(\tilde{\Psi}_0) + C/\rho^2$, and hence

$$\mathcal{E}'(\Psi_0)|_x \leq \fint_{B_{\rho/2}(x)} \mathcal{E}'(\Psi_0) + \frac{C}{\rho^2}. \tag{3.17}$$

It remains only to show that $\fint_{B_{\rho/2}(x)} \mathcal{E}'(\Psi_0)$ is of order $1/\rho^2$.

We divide this argument into two cases: away from a neighborhood of the punctures, and in a neighborhood of the punctures. First assume that x is away from any puncture. Then since U_0 is uniformly bounded, we can use the arguments of (10) and (11) in [25, pages 842-843] to obtain

$$\int_{B_{\rho/2}(x)} \mathcal{E}'(\Psi_0) \leq \left(4 \sup_{B_{3\rho/4}(x)} |U_0|^2 + 2 \sup_{B_{3\rho/4}(x)} |U_0|\right) \int_{B_{3\rho/4}(x)} |\nabla\varphi|^2, \tag{3.18}$$

where φ is a cut-off function which is 1 on $B_{\rho/2}(x)$ and vanishes outside $B_{3\rho/4}(x)$. Clearly $|\nabla\varphi|^2$ is of order $1/\rho^2$ in $B_{3\rho/4}(x)$, so after dividing by $\mathrm{vol}(B_{\rho/2}(x))$ the desired result follows.

Assume now that x is in a neighborhood of one of the punctures, which without loss of generality may be taken to be the origin. From the estimate (3.11), of U_0 near the puncture, we have $|U_0 - \log r| \leq C$. Furthermore $C^{-1}r(x) \leq r \leq Cr(x)$ on $B_{3\rho/4}(x)$, so that $|\log r - \log r(x)| \leq \log C$ on this domain. It follows that $|U_0 - \log r(x)| \leq C + \log C$. Since $\log r(x)$ is constant on $B_{3\rho/4}(x)$, we can replace U_0 by $U_0' = U_0 - \log r(x)$ in the arguments above to obtain

$$\int_{B_{\rho/2}(x)} \mathcal{E}'(\Psi_0) \leq \left(4 \sup_{B_{3\rho/4}(x)} |U_0'|^2 + 2 \sup_{B_{3\rho/4}(x)} |U_0'|\right) \int_{B_{3\rho/4}(x)} |\nabla\varphi|^2. \tag{3.19}$$

This completes the proof of (3.13). The estimates (3.14) follow immediately from (3.13) and (3.11). □

The previous proposition is useful because the bounds it provides in (3.13) apply globally. However, more detailed estimates are possible near any compact subset of the axis Γ which does not include punctures. More precisely, Ψ_0 is smooth away from the punctures and satisfies the following asymptotics [20]:

$$v_0 = c_1 + c_3\chi_0 - c_2\psi_0 + O(\rho^4) = c_4 + O(\rho^2), \qquad \chi_0 = c_2 + O(\rho^2), \qquad \psi_0 = c_3 + O(\rho^2), \tag{3.20}$$

$$|\nabla(U_0 - \log\rho)| = O(\rho^{-1+\epsilon}), \qquad |\Delta U_0| \leq C\rho^{-2+\epsilon}, \tag{3.21}$$

for some $\epsilon > 0$, and

$$|\nabla v_0| = O(\rho), \qquad |\nabla\chi_0| + |\nabla\psi_0| = O(\rho), \qquad |\nabla v_0 + \chi_0\nabla\psi_0 - \psi_0\nabla\chi_0| = O(\rho^3). \tag{3.22}$$

Lastly, we will have need of the following weighted Poincaré inequalities.

Lemma 3.4. *Let $\beta > \frac{1}{2}$ and $h \in C^0$. If $f \in C^1$ is axisymmetric and satisfies $(\sin^{-\beta}\theta)f|_{\theta=0,\pi} = 0$ then*

$$\int_{\delta_1<r<\delta_2} \frac{f^2}{\sin^{2\beta+1}\theta}\frac{h(r)}{r^2} \leq C\int_{\delta_1<r<\delta_2} \frac{|\nabla f|^2}{\sin^{2\beta-1}\theta}h(r) \tag{3.23}$$

where we are using r to denote the Euclidean distance to any puncture.

If $f \in C^1$ is an axisymmetric function with $f|_{r=\delta_1,\delta_2} = 0$, and $\beta \neq 0$ then

$$\int_{\delta_1<r<\delta_2} \frac{f^2}{r^{2\beta+2}} \leq C|\beta|^{-2} \int_{\delta_1<r<\delta_2} \frac{(\partial_r f)^2}{r^{2\beta}}. \tag{3.24}$$

Proof. The first statement slightly generalizes Proposition 2.4 in [9], but the proof there easily extends to this situation by keeping track of boundary terms.

For the second statement, project everything to the (ρ, z) plane so that the integration takes place over an annulus $A(\delta_1, \delta_2)$. Now use the fact that $\log r$ is harmonic in two dimensions, and the fact that $f|_{r=\delta_1,\delta_2} = 0$ to find

$$0 = \int_{A(\delta_1,\delta_2)} \nabla(\log r) \cdot \nabla(r^{-2\beta} f^2) = \int_{A(\delta_1,\delta_2)} \frac{1}{r} \left(-2\beta r^{-2\beta-1} f^2 + 2r^{-2\beta} f \partial_r f \right). \tag{3.25}$$

From this we easily get the desired inequality. □

4. Convexity and Proof of Theorem 1.2

Let $\tilde{\Psi} = (u, v, \chi, \psi) : \mathbb{R}^3 \to \mathbb{H}^2_{\mathbb{C}}$ and consider the harmonic energy on a domain $\Omega \subset \mathbb{R}^3$:

$$E_\Omega(\tilde{\Psi}) = \int_\Omega |\nabla u|^2 + e^{4u}|\nabla v + \chi\nabla\psi - \psi\nabla\chi|^2 + e^{2u} \left(|\nabla\chi|^2 + |\nabla\psi|^2\right) dx. \tag{4.1}$$

If Ω does not intersect the rotation axis $\Gamma = \{\rho = 0\}$, and we write $U = u + \log\rho$, then the reduced energy $\mathcal{I}_\Omega$ of the map $\Psi = (U, v, \chi, \psi)$ is related to the harmonic energy of $\tilde{\Psi}$ by

$$\mathcal{I}_\Omega(\Psi) = E_\Omega(\tilde{\Psi}) + \int_{\partial\Omega} (2U - \log\rho)\partial_\nu \log\rho, \tag{4.2}$$

where ν denotes the unit outer normal to the boundary $\partial\Omega$ and

$$\mathcal{I}_\Omega(\Psi) = \int_\Omega |\nabla U|^2 + \frac{e^{4U}}{\rho^4}|\nabla v + \chi\nabla\psi - \psi\nabla\chi|^2 + \frac{e^{2U}}{\rho^2} \left(|\nabla\chi|^2 + |\nabla\psi|^2\right) dx. \tag{4.3}$$

The formula (4.2) is obtained through an integration by parts, using the fact that $\log\rho$ is harmonic on $\mathbb{R}^3 \setminus \Gamma$. Note that $\mathcal{I} = \mathcal{I}_{\mathbb{R}^3} = 8\pi\mathcal{M}$ where $\mathcal{M}$ was introduced in Section 1. Moreover $\mathcal{I}$, which is referred to as the reduced energy, may be considered a regularization of E since the infinite term $\int |\nabla \log\rho|^2$ has been removed, and since the two functionals differ only by a boundary term they must have the same critical points.

Let $\tilde{\Psi}_0 = (u_0, v_0, \chi_0, \psi_0)$ denote the harmonic map constructed in the previous section, and let $\Psi_0 = (U_0, v_0, \chi_0, \psi_0)$ be the associated renormalized map where $U_0 = u_0 + \log\rho$. Thus, Ψ_0 is a critical point of $\mathcal{I}$. It is the purpose of this section to show that Ψ_0 realizes the global minimum for $\mathcal{I}$.

Theorem 4.1. *Suppose that $\Psi = (U, v, \chi, \psi)$ is smooth and satisfies the asymptotics* (1.8)-(1.10), (4.8)-(4.11) *with $v|_\Gamma = v_0|_\Gamma$, $\chi|_\Gamma = \chi_0|_\Gamma$, $\psi|_\Gamma = \psi_0|_\Gamma$, then there exists a constant $C > 0$ such that*

$$\mathcal{I}(\Psi) - \mathcal{I}(\Psi_0) \geq C \left(\int_{\mathbb{R}^3} \text{dist}^6_{\mathbb{H}^2_{\mathbb{C}}}(\Psi, \Psi_0) dx \right)^{\frac{1}{3}}. \tag{4.4}$$

This theorem is analogous to that of Theorem 6.1 in [22], where the role of extreme Kerr-Newman is now played by the (possibly) multiple black hole solution Ψ_0 constructed in the previous section. The proof in [22] is based on convexity of the harmonic energy under geodesic deformations; such a property is true under general circumstances when the target space is nonpositively curved. More precisely, let $\delta, \varepsilon > 0$ be small parameters and set $\Omega_{\delta,\varepsilon} = \{\delta < r_n \text{ for } n = 1, \ldots, N; r < 2/\delta; \rho > \varepsilon\}$

and $\mathcal{A}_{\delta,\varepsilon} = B_{2/\delta} \backslash \Omega_{\delta,\varepsilon}$, where $B_{2/\delta}$ is the ball of radius $2/\delta$ centered at the origin. Via a cut-and-paste argument, it will be shown that we may assume Ψ satisfies

$$\mathrm{supp}(U - U_0) \subset B_{2/\delta}, \qquad \mathrm{supp}(v - v_0, \chi - \chi_0, \psi - \psi_0) \subset \Omega_{\delta,\varepsilon}. \tag{4.5}$$

If $\tilde{\Psi}_t$, $t \in [0,1]$, is a geodesic in $\mathbb{H}^2_{\mathbb{C}}$ connecting $\tilde{\Psi}_1 = \tilde{\Psi}$ and $\tilde{\Psi}_0$ (this means that for each x in the domain, $t \to \tilde{\Psi}_t(x)$ is a geodesic), then $\Psi_t \equiv \Psi_0$ outside $B_{2/\delta}$ and $(v_t, \chi_t, \psi_t) \equiv (v_0, \chi_0, \psi_0)$ in a neighborhood of $\mathcal{A}_{\delta,\varepsilon}$, so that in particular $U_t = U_0 + t(U - U_0)$ on these domains. This simple expression for U_t together with convexity of the harmonic energy yields

$$\frac{d^2}{dt^2}\mathcal{I}(\Psi_t) \geq 2\int_{\mathbb{R}^3} |\nabla \,\mathrm{dist}_{\mathbb{H}^2_{\mathbb{C}}}(\Psi, \Psi_0)|^2 dx. \tag{4.6}$$

Moreover, the fact that Ψ_0 is a critical point implies that

$$\frac{d}{dt}\mathcal{I}(\Psi_t)|_{t=0} = 0. \tag{4.7}$$

Theorem 4.1 then follows by integrating (4.6) and applying a Sobolev inequality. In the remainder of this section we will justify each of these steps, following closely the strategy of [22] in the case of a single black hole. Most of the effort required to establish each step consists of estimating certain integrals. Here, however, the techniques used for these estimates will be significantly different since Ψ_0 is not known explicitly, whereas in the single black hole case Ψ_0 is explicit as it arises from the extreme Kerr-Newman solution.

Before proceeding we record the appropriate asymptotic behavior of Ψ. Asymptotics for U are given in (1.8), (1.9), (1.10), and if $\omega = dv + \chi d\psi - \psi d\chi$ then

$$|\omega| = \rho^2 O(r^{-\lambda}), \qquad |\nabla\chi| + |\nabla\psi| = \rho O(r^{-\lambda}) \quad \text{as} \quad r \to \infty, \tag{4.8}$$

$$|\omega| = \rho^2 O(r_n^{\lambda-6}), \quad |\nabla\chi| + |\nabla\psi| = \rho O(r_n^{\lambda-4}) \quad \text{as} \quad r_n \to 0 \quad \text{in asymptotically flat ends}, \tag{4.9}$$

$$|\omega| = \rho^2 O(r_n^{\lambda-5}), \quad |\nabla\chi| + |\nabla\psi| = \rho O(r_n^{\lambda-3}) \quad \text{as} \quad r_n \to 0 \quad \text{in asymptotically cylindrical ends}, \tag{4.10}$$

$$|\omega| = O(\rho^2), \qquad |\nabla\chi| + |\nabla\psi| = O(\rho) \quad \text{as} \quad \rho \to 0 \quad \text{in} \quad \Omega_{\delta,\varepsilon}. \tag{4.11}$$

Note that with these asymptotics $\mathcal{I}(\Psi)$ is finite precisely when $\lambda > \frac{3}{2}$. Moreover, one may integrate along lines perpendicular to Γ to find

$$|\chi|, |\psi| = \text{const} + \rho^2 O(r^{-\lambda}) \quad \text{as} \quad r \to \infty, \tag{4.12}$$

$$|\chi|, |\psi| = \text{const} + \rho^2 O(r_n^{\lambda-4}) \quad \text{as} \quad r_n \to 0 \quad \text{in asymptotically flat ends}, \tag{4.13}$$

$$|\chi|, |\psi| = \text{const} + \rho^2 O(r_n^{\lambda-3}) \quad \text{as} \quad r_n \to 0 \quad \text{in asymptotically cylindrical ends}, \tag{4.14}$$

$$|\chi|, |\psi| = \text{const} + O(\rho^2) \quad \text{as} \quad \rho \to 0 \quad \text{in} \quad \Omega_{\delta,\varepsilon}, \tag{4.15}$$

from which it follows that

$$|\nabla v| = \rho O(r^{-\lambda+1}) \quad \text{as} \quad r \to \infty, \tag{4.16}$$

$$|\nabla v| = \rho O(r_n^{\lambda-5}) \quad \text{as} \quad r_n \to 0 \quad \text{in asymptotically flat ends}, \tag{4.17}$$

$$|\nabla v| = \rho O(r_n^{\lambda-4}) \quad \text{as} \quad r_n \to 0 \quad \text{in asymptotically cylindrical ends}, \tag{4.18}$$

$$|\nabla v| = \rho O(r^{-\lambda+1}) \quad \text{as} \quad \rho \to 0 \quad \text{in} \quad \Omega_{\delta,\varepsilon}. \tag{4.19}$$

In order to carry out the proof of Theorem 4.1 as outlined above, we must first show that it is possible to approximate $\mathcal{I}(\Psi)$ by replacing Ψ with a map that satisfies (4.5). This may be achieved as in [22] with a three step cut and paste argument. Define smooth cut-off functions

$$\varphi_\delta^1 = \begin{cases} 1 & \text{if } r \leq \frac{1}{\delta}, \\ |\nabla \varphi_\delta^1| \leq 2\delta & \text{if } \frac{1}{\delta} < r < \frac{2}{\delta}, \\ 0 & \text{if } r \geq \frac{2}{\delta}, \end{cases} \tag{4.20}$$

$$\varphi_\delta = \begin{cases} 0 & \text{if } r_n \leq \delta, \\ |\nabla \varphi_\delta| \leq \frac{2}{\delta} & \text{if } \delta < r_n < 2\delta, \\ 1 & \text{if } r_n \geq 2\delta, \end{cases} \quad \text{for} \quad n = 1, \dots, N, \tag{4.21}$$

$$\phi_\varepsilon = \begin{cases} 0 & \text{if } \rho \leq \varepsilon, \\ \frac{\log(\rho/\varepsilon)}{\log(\sqrt{\varepsilon}/\varepsilon)} & \text{if } \varepsilon < \rho < \sqrt{\varepsilon}, \\ 1 & \text{if } \rho \geq \sqrt{\varepsilon}. \end{cases} \tag{4.22}$$

The first step deals with the region M^0_{end}. Let

$$F_\delta^1(\Psi) = \Psi_0 + \varphi_\delta^1(\Psi - \Psi_0) =: (U_\delta^1, v_\delta^1, \chi_\delta^1, \psi_\delta^1), \tag{4.23}$$

so that $F_\delta^1(\Psi) = \Psi_0$ on $\mathbb{R}^3 \setminus B_{2/\delta}$.

Lemma 4.2. $\lim_{\delta \to 0} \mathcal{I}(F_\delta^1(\Psi)) = \mathcal{I}(\Psi)$.

Proof. Write

$$\mathcal{I}(F_\delta^1(\Psi)) = \mathcal{I}_{r \leq \frac{1}{\delta}}(F_\delta^1(\Psi)) + \mathcal{I}_{\frac{1}{\delta} < r < \frac{2}{\delta}}(F_\delta^1(\Psi)) + \mathcal{I}_{r \geq \frac{2}{\delta}}(F_\delta^1(\Psi)), \tag{4.24}$$

and observe that $\mathcal{I}_{r \leq \frac{1}{\delta}}(F_\delta^1(\Psi)) \to \mathcal{I}(\Psi)$ by the dominated convergence theorem (DCT) and since Ψ_0 has finite reduced energy $\mathcal{I}_{r \geq \frac{2}{\delta}}(F_\delta^1(\Psi)) \to 0$. Now write

$$\mathcal{I}_{\frac{1}{\delta} < r < \frac{2}{\delta}}(F_\delta^1(\Psi)) = \underbrace{\int_{\frac{1}{\delta} < r < \frac{2}{\delta}} |\nabla U_\delta^1|^2}_{I_1} + \underbrace{\int_{\frac{1}{\delta} < r < \frac{2}{\delta}} \frac{e^{4U_\delta^1}}{\rho^4} |\omega_\delta^1|^2}_{I_2} + \underbrace{\int_{\frac{1}{\delta} < r < \frac{2}{\delta}} \frac{e^{2U_\delta^1}}{\rho^2} (|\nabla \chi_\delta^1|^2 + |\nabla \psi_\delta^1|^2)}_{I_3}. \tag{4.25}$$

We have

$$I_1 \leq 2 \int_{\frac{1}{\delta} < r < \frac{2}{\delta}} \left(|\nabla U|^2 + |\nabla U_0|^2 + (U - U_0)^2 \underbrace{|\nabla \varphi_\delta^1|^2}_{\leq 4\delta^2} \right), \tag{4.26}$$

where the first two terms converge to zero by the DCT and finite reduced energy of Ψ_0, respectively. For the third term we may apply Hölder's inequality and the Gagliardo-Nirenberg-Sobolev inequality to find

(4.27)

$$\int_{\frac{1}{\delta} < r < \frac{2}{\delta}} (U - U_0)^2 \underbrace{|\nabla \varphi_\delta^1|^2}_{\leq 4\delta^2} \leq \left(\int_{\frac{1}{\delta} < r < \frac{2}{\delta}} (U - U_0)^6 \right)^{\frac{1}{3}} \left(\int_{\frac{1}{\delta} < r < \frac{2}{\delta}} |\nabla \varphi_\delta^1|^3 \right)^{\frac{2}{3}} \leq C \int_{\frac{1}{\delta} < r < \frac{2}{\delta}} |\nabla (U - U_0)|^2 \to 0.$$

Note that the Gagliardo-Nirenberg-Sobolev inequality applies here since $U, U_0 \in H^1(\mathbb{R}^3)$ (the Sobolev space of square integrable derivatives) are limits of compactly supported functions.

Now consider I_2, and write

$$\begin{aligned}\omega^1_\delta =&\varphi^1_\delta\omega+(1-\varphi^1_\delta)\omega_0+(v-v_0)\nabla\varphi^1_\delta+(\chi_0\psi-\psi_0\chi)\nabla\varphi^1_\delta\\ &+\varphi^1_\delta(1-\varphi^1_\delta)[(\psi-\psi_0)\nabla(\chi-\chi_0)-(\chi-\chi_0)\nabla(\psi-\psi_0)].\end{aligned} \tag{4.28}$$

Using (3.12) and (4.20) produces

$$\begin{aligned}I_2 \leq &C\int_{\frac{1}{\delta}<r<\frac{2}{\delta}}\rho^{-4}(|\omega|^2+|\omega_0|^2+r^{-2}|v-v_0|^2+r^{-2}|\chi_0\psi-\psi_0\chi|^2\\ &+|\psi-\psi_0|^2|\nabla(\chi-\chi_0)|^2+|\chi-\chi_0|^2|\nabla(\psi-\psi_0)|^2).\end{aligned} \tag{4.29}$$

The first and second terms converge to zero by the DCT and finite reduced energy of Ψ_0. Next, by (3.20) and (4.19) we have $\rho^{-\frac{3}{2}}|v-v_0|\to 0$ as $\rho\to 0$ so that Lemma 3.4 applies, together with (3.12) to show

$$\begin{aligned}\int_{\frac{1}{\delta}<r<\frac{2}{\delta}}\frac{|v-v_0|^2}{r^2\rho^4} \leq &C\int_{\frac{1}{\delta}<r<\frac{2}{\delta}}\frac{|\nabla(v-v_0)|^2}{r^2\rho^2}\\ \leq &C\int_{\frac{1}{\delta}<r<\frac{2}{\delta}}\frac{e^{4U}}{\rho^4}|\omega|^2+\frac{e^{4U_0}}{\rho^4}|\omega_0|^2\\ &+C\int_{\frac{1}{\delta}<r<\frac{2}{\delta}}\frac{e^{2U}}{r^2\rho^2}(|\nabla\chi|^2+|\nabla\psi|^2)+\frac{e^{2U_0}}{r^2\rho^2}(|\nabla\chi_0|^2+|\nabla\psi_0|^2)\\ &\to 0,\end{aligned} \tag{4.30}$$

where the DCT and finite reduced energy were used in the last step. A similar argument holds for the fourth term on the right-hand side of (4.29), whereas the fifth and sixth terms may be directly estimated by terms in the reduced energy of Ψ and Ψ_0. It follows that $I_2\to 0$.

Consider the first term in the integral I_3 and write

$$\nabla\chi^1_\delta=\nabla\chi_0+(\chi-\chi_0)\nabla\varphi^1_\delta+\varphi^1_\delta\nabla(\chi-\chi_0), \tag{4.31}$$

so that

$$\int_{\frac{1}{\delta}<r<\frac{2}{\delta}}\frac{e^{2U^1_\delta}}{\rho^2}|\nabla\chi^1_\delta|^2\leq C\int_{\frac{1}{\delta}<r<\frac{2}{\delta}}\left(\frac{|\nabla\chi_0|^2}{\rho^2}+\frac{|\chi-\chi_0|^2}{r^2\rho^2}+\frac{|\nabla(\chi-\chi_0)|^2}{\rho^2}\right). \tag{4.32}$$

The first and third terms on the right-hand side may be estimated in terms of the reduced energy. The same is true of the second term, after an application of Lemma 3.4. Since similar considerations hold for the second term in I_3, it follows that $I_3\to 0$. □

Consider now small balls centered at the punctures p_n. Let

$$F_\delta(\Psi)=(U,v_\delta,\chi_\delta,\psi_\delta) \tag{4.33}$$

where

$$(v_\delta,\chi_\delta,\psi_\delta)=(v_0,\chi_0,\psi_0)+\varphi_\delta(v-v_0,\chi-\chi_0,\psi-\psi_0), \tag{4.34}$$

so that $F_\delta(\Psi)=\Psi_0$ on $\cup_{n=1}^N B_\delta(p_n)$.

Lemma 4.3. $\lim_{\delta\to 0}\mathcal{I}(F_\delta(\Psi))=\mathcal{I}(\Psi)$. *This also holds if* $\Psi\equiv\Psi_0$ *outside* $B_{2/\delta}$.

Proof. Write

$$\mathcal{I}(F_\delta(\Psi)) = \sum_{n=1}^{N} \left[\mathcal{I}_{r_n\leq\delta}(F_\delta(\Psi)) + \mathcal{I}_{\delta<r_n<2\delta}(F_\delta(\Psi)) + \mathcal{I}_{r_n\geq 2\delta}(F_\delta(\Psi))\right], \tag{4.35}$$

and observe that by DCT

$$\sum_{n=1}^{N} \mathcal{I}_{r_n\geq 2\delta}(F_\delta(\Psi)) = \sum_{n=1}^{N} \mathcal{I}_{r_n\geq 2\delta}(\Psi) \to \mathcal{I}(\Psi). \tag{4.36}$$

Moreover

$$\mathcal{I}_{r_n\leq\delta}(F_\delta(\Psi)) = \int_{r_n\leq\delta} |\nabla U|^2 + \frac{e^{4U}}{\rho^4}|\omega_0|^2 + \frac{e^{2U}}{\rho^2}(|\nabla\chi_0|^2 + |\nabla\psi_0|^2), \tag{4.37}$$

where the first term on the right-hand side converges to zero again by DCT. The second and third terms may be estimated by the reduced energy of Ψ_0 (and hence also converge to zero), since the asymptotics (1.9), (1.10), and (3.11) imply that

$$e^U \leq c e^{U_0} \tag{4.38}$$

near each puncture.

Now write

$$\mathcal{I}_{\delta<r_n<2\delta}(F_\delta(\Psi)) = \underbrace{\int_{\delta<r_n<2\delta} |\nabla U|^2}_{I_1} + \underbrace{\int_{\delta<r_n<2\delta} \frac{e^{4U}}{\rho^4}|\omega_\delta|^2}_{I_2} + \underbrace{\int_{\delta<r_n<2\delta} \frac{e^{2U}}{\rho^2}(|\nabla\chi_\delta|^2 + |\nabla\psi_\delta|^2)}_{I_3}, \tag{4.39}$$

and observe that $I_1 \to 0$ by DCT. In order to estimate I_2, write

$$\begin{aligned}\omega_\delta =& \varphi_\delta\omega + (1-\varphi_\delta)\omega_0 + (v - v_0)\nabla\varphi_\delta + (\chi_0\psi - \psi_0\chi)\nabla\varphi_\delta \\ &+ \varphi_\delta(1-\varphi_\delta)[(\psi-\psi_0)\nabla(\chi-\chi_0) - (\chi-\chi_0)\nabla(\psi-\psi_0)].\end{aligned} \tag{4.40}$$

Using (4.21) and (4.38) produces

$$\begin{aligned} I_2 \leq& C\int_{\delta<r_n<2\delta} \left(\frac{e^{4U}}{\rho^4}|\omega|^2 + \frac{e^{4U_0}}{\rho^4}|\omega_0|^2 + \frac{e^{4U}}{r_n^2\rho^4}|v - v_0|^2 + \frac{e^{4U}}{r_n^2\rho^4}|\chi_0\psi - \psi_0\chi|^2\right) \\ &+ C\int_{\delta<r_n<2\delta} \frac{e^{4U}}{\rho^4}\left(|\psi-\psi_0|^2|\nabla(\chi-\chi_0)|^2 + |\chi-\chi_0|^2|\nabla(\psi-\psi_0)|^2\right). \end{aligned} \tag{4.41}$$

The first and second terms converge to zero by the DCT and finite reduced energy of Ψ_0.

Next, assume that p_n represents a cylindrical end, so that both $U, U_0 \sim \log r_n$. By (3.20) and (4.19) we have $\rho^{-\frac{3}{2}}|v - v_0| \to 0$ as $\rho \to 0$ so that Lemma 3.4 applies to yield

$$\begin{aligned} \int_{\delta<r_n<2\delta} \frac{e^{4U}}{r_n^2\rho^4}|v - v_0|^2 \leq& C\int_{\delta<r_n<2\delta} \frac{r_n^2}{\rho^4}|v - v_0|^2 \\ \leq& C\int_{\delta<r_n<2\delta} \frac{r_n^2}{\rho^2}|\nabla(v - v_0)|^2 \\ \leq& C\int_{\delta<r_n<2\delta} \frac{e^{4U}}{\rho^4}|\omega|^2 + \frac{e^{4U_0}}{\rho^4}|\omega_0|^2 \\ &+ C\int_{\delta<r_n<2\delta} \frac{e^{2U}}{\rho^2}(|\nabla\chi|^2 + |\nabla\psi|^2) + \frac{e^{2U_0}}{\rho^2}(|\nabla\chi_0|^2 + |\nabla\psi_0|^2) \\ &\to 0, \end{aligned} \tag{4.42}$$

where the DCT and finite reduced energy were used in the last step. A similar argument holds for the fourth term on the right-hand side of (4.41). The fifth and sixth terms may be directly estimated by terms in the reduced energy of Ψ and Ψ_0, since

$$\frac{|\chi-\chi_0|}{\sin\theta}+\frac{|\psi-\psi_0|}{\sin\theta}\leq C. \tag{4.43}$$

To establish this, observe that by the mean value theorem

$$|(\chi-\chi_0)(r_n,\theta)|=\theta|\partial_\theta(\chi-\chi_0)(r_n,\theta')|\leq Cr_n\theta(|\nabla\chi(r_n,\theta')|+|\nabla\chi_0(r_n,\theta')|)\leq C\theta \tag{4.44}$$

for some $\theta'<\theta$, where we have used (3.14) and (4.10). Performing a similar calculation based at $\theta=\pi$, then yields (4.43), after noting that ψ,ψ_0 behave analogously to χ,χ_0. Lastly, in the case that p_n represents an asymptotically flat end, similar computations yield the desired result. It follows that $I_2\to 0$.

Consider the first term in the integral I_3 and write

$$\nabla\chi_\delta=\nabla\chi_0+(\chi-\chi_0)\nabla\varphi_\delta+\varphi_\delta\nabla(\chi-\chi_0), \tag{4.45}$$

so that

$$\int_{\delta<r_n<2\delta}\frac{e^{2U}}{\rho^2}|\nabla\chi_\delta|^2\leq C\int_{\delta<r_n<2\delta}\left(\frac{e^{2U_0}}{\rho^2}|\nabla\chi_0|^2+\frac{e^{2U}}{r_n^2\rho^2}|\chi-\chi_0|^2+\frac{e^{2U}}{\rho^2}|\nabla(\chi-\chi_0)|^2\right). \tag{4.46}$$

The first and third terms on the right-hand side may be estimated in terms of the reduced energy. The same is true of the second term, after an application of Lemma 3.4 as above. Since similar considerations hold for the second term in I_3, and it follows that $I_3\to 0$. □

Consider now cylindrical regions around the axis Γ and away from the punctures given by

$$\mathcal{C}_{\delta,\varepsilon}=\{\rho\leq\varepsilon\}\cap\{\delta\leq r_n \text{ for } n=1,\ldots,N; r\leq 2/\delta\}, \tag{4.47}$$

$$\mathcal{W}_{\delta,\varepsilon}=\{\varepsilon\leq\rho\leq\sqrt{\varepsilon}\}\cap\{\delta\leq r_n \text{ for } n=1,\ldots,N; r\leq 2/\delta\}. \tag{4.48}$$

Let

$$G_\varepsilon(\Psi)=(U,v_\varepsilon,\chi_\varepsilon,\psi_\varepsilon) \tag{4.49}$$

where

$$(v_\varepsilon,\chi_\varepsilon,\psi_\varepsilon)=(v_0,\chi_0,\psi_0)+\phi_\varepsilon(v-v_0,\chi-\chi_0,\psi-\psi_0), \tag{4.50}$$

so that $G_\varepsilon(\Psi)=\Psi_0$ on $\rho\leq\varepsilon$.

Lemma 4.4. *Fix $\delta>0$ and suppose that $\Psi\equiv\Psi_0$ on $\cup_{n=1}^N B_\delta(p_n)$, then $\lim_{\varepsilon\to 0}\mathcal{I}(G_\varepsilon(\Psi))=\mathcal{I}(\Psi)$. This also holds if $\Psi\equiv\Psi_0$ outside $B_{2/\delta}$.*

Proof. Write

$$\mathcal{I}(G_\varepsilon(\Psi))=\mathcal{I}_{\mathcal{C}_{\delta,\varepsilon}}(G_\varepsilon(\Psi))+\mathcal{I}_{\mathcal{W}_{\delta,\varepsilon}}(G_\varepsilon(\Psi))+\mathcal{I}_{\mathbb{R}^3\setminus(\mathcal{C}_{\delta,\varepsilon}\cup\mathcal{W}_{\delta,\varepsilon})}(G_\varepsilon(\Psi)). \tag{4.51}$$

Since $\Psi\equiv\Psi_0$ on $\cup_{n=1}^N B_\delta(p_n)$, the DCT and finite energy of Ψ_0 imply that

$$\mathcal{I}_{\mathbb{R}^3\setminus(\mathcal{C}_{\delta,\varepsilon}\cup\mathcal{W}_{\delta,\varepsilon})}(G_\varepsilon(\Psi))\to\mathcal{I}(\Psi). \tag{4.52}$$

Moreover

$$\mathcal{I}_{\mathcal{C}_{\delta,\varepsilon}}(G_\varepsilon(\Psi))=\int_{\mathcal{C}_{\delta,\varepsilon}}|\nabla U|^2+\frac{e^{4U}}{\rho^4}|\omega_0|^2+\frac{e^{2U}}{\rho^2}(|\nabla\chi_0|^2+|\nabla\psi_0|^2), \tag{4.53}$$

where the first term on the right-hand side converges to zero again by DCT. The second and third terms may be estimated by the reduced energy of Ψ_0 (and hence also converge to zero), since

$$|U| + |U_0| \le C \quad \text{on} \quad \mathbb{R}^3 \setminus \cup_{n=1}^N B_\delta(p_n) \tag{4.54}$$

by (3.12).

Now write

$$\mathcal{I}_{\mathcal{W}_{\delta,\varepsilon}}(G_\varepsilon(\Psi)) = \underbrace{\int_{\mathcal{W}_{\delta,\varepsilon}} |\nabla U|^2}_{I_1} + \underbrace{\int_{\mathcal{W}_{\delta,\varepsilon}} \frac{e^{4U}}{\rho^4}|\omega_\varepsilon|^2}_{I_2} + \underbrace{\int_{\mathcal{W}_{\delta,\varepsilon}} \frac{e^{2U}}{\rho^2}(|\nabla\chi_\varepsilon|^2 + |\nabla\psi_\varepsilon|^2)}_{I_3}, \tag{4.55}$$

and notice that $I_1 \to 0$ by DCT. In order to estimate I_2, write

$$\begin{aligned} \omega_\varepsilon =& \phi_\varepsilon\omega + (1-\phi_\varepsilon)\omega_0 + (v - v_0)\nabla\phi_\varepsilon + (\chi_0\psi - \psi_0\chi)\nabla\phi_\varepsilon \\ &+ \phi_\varepsilon(1-\phi_\varepsilon)[(\psi-\psi_0)\nabla(\chi-\chi_0) - (\chi-\chi_0)\nabla(\psi-\psi_0)]. \end{aligned} \tag{4.56}$$

Using (4.22) and (4.54) produces

$$\begin{aligned} I_2 \le& C\int_{\mathcal{W}_{\delta,\varepsilon}} \rho^{-4}(|\omega|^2 + |\omega_0|^2 + (\log\varepsilon)^{-2}\rho^{-2}|v-v_0|^2 + (\log\varepsilon)^{-2}\rho^{-2}|\chi_0\psi - \psi_0\chi|^2 \\ &+ |\psi-\psi_0|^2|\nabla(\chi-\chi_0)|^2 + |\chi-\chi_0|^2|\nabla(\psi-\psi_0)|^2). \end{aligned} \tag{4.57}$$

The first and second terms converge to zero by the DCT and finite reduced energy of Ψ_0. The third term may be directly estimated with the help of (3.20) and (4.19)

$$\int_{\mathcal{W}_{\delta,\varepsilon}} (\log\varepsilon)^{-2}\rho^{-6}|v-v_0|^2 \le C\int_{\mathcal{W}_{\delta,\varepsilon}} (\log\varepsilon)^{-2}\rho^{-2} \le C(\log\varepsilon)^{-1} \to 0. \tag{4.58}$$

A similar calculation holds for the fourth term. Consider now the fifth term, and use (3.20), (4.11), and (4.15) to find

$$\int_{\mathcal{W}_{\delta,\varepsilon}} \rho^{-4}|\psi-\psi_0|^2|\nabla(\chi-\chi_0)|^2 \le C\int_{\mathcal{W}_{\delta,\varepsilon}} \rho^2 \le C\varepsilon^2 \to 0. \tag{4.59}$$

The sixth term behaves in the same way, and hence $I_2 \to 0$.

Consider the first term in the integral I_3 and write

$$\nabla\chi_\varepsilon = \nabla\chi_0 + (\chi-\chi_0)\nabla\phi_\varepsilon + \phi_\varepsilon\nabla(\chi-\chi_0), \tag{4.60}$$

so that

$$\int_{\mathcal{W}_{\delta,\varepsilon}} \frac{e^{2U}}{\rho^2}|\nabla\chi_\varepsilon|^2 \le C\int_{\mathcal{W}_{\delta,\varepsilon}} \rho^{-2}(|\nabla\chi_0|^2 + (\log\varepsilon)^{-2}\rho^{-2}|\chi-\chi_0|^2 + |\nabla(\chi-\chi_0)|^2). \tag{4.61}$$

All of these terms may be estimated as above, showing that $I_3 \to 0$. □

By composing the three cut and paste operations defined above, we obtain the desired replacement for Ψ which satisfies (4.5). Namely, let

$$\Psi_{\delta,\varepsilon} = G_\varepsilon\left(F_\delta\left(F_\delta^1(\Psi)\right)\right). \tag{4.62}$$

Proposition 4.5. *Let $\varepsilon \ll \delta \ll 1$ and suppose that Ψ satisfies the hypotheses of Theorem 4.1. Then $\Psi_{\delta,\varepsilon}$ satisfies* (4.5) *and*

$$\lim_{\delta\to0}\lim_{\varepsilon\to0}\mathcal{I}(\Psi_{\delta,\varepsilon}) = \mathcal{I}(\Psi). \tag{4.63}$$

We are now in a position to prove the main result of this section.

Proof of Theorem 4.1. By Proposition 4.5 $\Psi_{\delta,\varepsilon}$ satisfies (4.5). Thus, if $\tilde{\Psi}^t_{\delta,\varepsilon}$ is the geodesic connecting $\tilde{\Psi}_0$ to $\tilde{\Psi}_{\delta,\varepsilon}$ as described at the beginning of this section, then $U^t_{\delta,\varepsilon} = U_0 + t(U_{\delta,\varepsilon} - U_0)$. Following [22] we have

$$\frac{d^2}{dt^2}\mathcal{I}(\Psi^t_{\delta,\varepsilon}) = \underbrace{\frac{d^2}{dt^2}\mathcal{I}_{\Omega_{\delta,\varepsilon}}(\Psi^t_{\delta,\varepsilon})}_{I_1} + \underbrace{\frac{d^2}{dt^2}\mathcal{I}_{\mathcal{A}_{\delta,\varepsilon}}(\Psi^t_{\delta,\varepsilon})}_{I_2}, \tag{4.64}$$

with

$$\begin{aligned} I_1 = & \frac{d^2}{dt^2}E_{\Omega_{\delta,\varepsilon}}(\tilde{\Psi}^t_{\delta,\varepsilon}) + \frac{d^2}{dt^2}\int_{\partial\Omega_{\delta,\varepsilon}\cap\partial\mathcal{A}_{\delta,\varepsilon}} (\partial_\nu \log\rho)\left[2(U_0 + t(U_{\delta,\varepsilon} - U_0)) - \log\rho\right] \\ \geq & 2\int_{\Omega_{\delta,\varepsilon}} |\nabla \operatorname{dist}_{\mathbb{H}^2_{\mathbb{C}}}(\Psi_{\delta,\varepsilon}, \Psi_0)|^2 \end{aligned} \tag{4.65}$$

where convexity of the harmonic energy [22] was used in the last step, and

$$\begin{aligned} I_2 = & \int_{\mathcal{A}_{\delta,\varepsilon}} 2|\nabla(U_{\delta,\varepsilon} - U_0)|^2 + 16(U_{\delta,\varepsilon} - U_0)^2 \frac{e^{4[U_0 + t(U_{\delta,\varepsilon} - U_0)]}}{\rho^4}|\omega_0|^2 \\ & + \int_{\mathcal{A}_{\delta,\varepsilon}} 4(U_{\delta,\varepsilon} - U_0)^2 \frac{e^{2[U_0 + t(U_{\delta,\varepsilon} - U_0)]}}{\rho^2}(|\nabla\chi_0|^2 + |\nabla\psi_0|^2) \\ \geq & 2\int_{\mathcal{A}_{\delta,\varepsilon}} |\nabla \operatorname{dist}_{\mathbb{H}^2_{\mathbb{C}}}(\Psi_{\delta,\varepsilon}, \Psi_0)|^2 \end{aligned} \tag{4.66}$$

since $\operatorname{dist}_{\mathbb{H}^2_{\mathbb{C}}}(\Psi_{\delta,\varepsilon}, \Psi_0) = |U_{\delta,\varepsilon} - U_0|$ on $\mathcal{A}_{\delta,\varepsilon}$.

It remains to show that passing $\frac{d^2}{dt^2}$ into the integral in (4.66) is justified. For this it is sufficient to show that each term on the right-hand side of the equality in (4.66) is uniformly integrable. There is no issue with the first term since $U_{\delta,\varepsilon}, U_0 \in H^1(\mathbb{R}^3)$. Consider now the second and third terms, and write $\mathcal{A}_{\delta,\varepsilon} = \mathcal{C}_{\delta,\varepsilon} \cup_{n=1}^N B_\delta(p_n)$. Uniform integrability will follow if $(U_{\delta,\varepsilon} - U_0)^2 e^{at(U_{\delta,\varepsilon} - U_0)}$, $a = 2, 4$ is uniformly bounded, since then these terms may be estimated by the reduced energy of Ψ_0. This is clearly the case on $\mathcal{C}_{\delta,\varepsilon}$, as U and U_0 are bounded on this region. On $B_\delta(p_n)$, $U_{\delta,\varepsilon} - U_0 \sim \log r_n$ if p_n represents an asymptotically flat end and $U_{\delta,\varepsilon} - U_0 \sim 1$ if p_n represents an asymptotically cylindrical end. Thus, the desired conclusion follows if $r_n^{at}(\log r_n)^2$ is uniformly bounded, which occurs for $0 < t_0 < t \leq 1$. Since $t_0 > 0$ is arbitrary, we conclude that (4.6) holds for $\Psi_{\delta,\varepsilon}$ when $t \in (0, 1]$.

We now aim to verify (4.7) for $\Psi_{\delta,\varepsilon}$. Choose $\varepsilon_0 < \varepsilon$, $\delta_0 < \delta$ and write

$$\frac{d}{dt}\mathcal{I}(\Psi^t_{\delta,\varepsilon}) = \underbrace{\frac{d}{dt}\mathcal{I}_{\Omega_{\delta_0,\varepsilon_0}}(\Psi^t_{\delta,\varepsilon})}_{I_3} + \underbrace{\frac{d}{dt}\mathcal{I}_{\mathcal{A}_{\delta_0,\varepsilon_0}}(\Psi^t_{\delta,\varepsilon})}_{I_4}. \tag{4.67}$$

Justification for passing $\frac{d}{dt}$ into the integrals, for $t \in (0, 1]$, is similar to the arguments of the previous paragraph. Then integrating by parts, and using the Euler-Lagrange equations (B.7) satisfied by Ψ_0 together with the fact that the functionals $\mathcal{I}$ and E have the same critical points, produces

$$I_3 = O(t) - \sum_{n=1}^N \int_{\partial B_{\delta_0}(p_n)} 2(U_{\delta,\varepsilon} - U_0)\partial_\nu U_0 - \int_{\partial\mathcal{C}_{\delta_0,\varepsilon_0}} 2(U_{\delta,\varepsilon} - U_0)\partial_\nu U_0 \tag{4.68}$$

for small t, where ν is the unit outer normal pointing toward M^0_{end}. Next, using that $U^t_{\delta,\varepsilon} = U_0 + t(U_{\delta,\varepsilon} - U_0)$ and $\frac{d}{dt} v^t_{\delta,\varepsilon} = \frac{d}{dt}\chi^t_{\delta,\varepsilon} = \frac{d}{dt}\psi^t_{\delta,\varepsilon} = 0$ yields

$$\begin{aligned} I_4 =& O(t) + \int_{\mathcal{A}_{\delta_0,\varepsilon_0}} 2\nabla U_0 \cdot \nabla(U_{\delta,\varepsilon} - U_0) + 4(U_{\delta,\varepsilon} - U_0)\frac{e^{4[U_0+t(U_{\delta,\varepsilon}-U_0)]}}{\rho^4}|\omega_0|^2 \\ &+ \int_{\mathcal{A}_{\delta_0,\varepsilon_0}} 2(U_{\delta,\varepsilon} - U_0)\frac{e^{2[U_0+t(U_{\delta,\varepsilon}-U_0)]}}{\rho^2}(|\nabla\chi_0|^2 + |\nabla\psi_0|^2). \end{aligned} \tag{4.69}$$

Observe that according to the first Euler-Lagrange equation of (B.7)

$$\begin{aligned} \int_{\partial B_{\delta_0}(p_n)} (U_{\delta,\varepsilon} - U_0)\partial_\nu U_0 =& \int_{B_{\delta_0}(p_n)} \nabla U_0 \cdot \nabla(U_{\delta,\varepsilon} - U_0) + 2(U_{\delta,\varepsilon} - U_0)\frac{e^{4U_0}}{\rho^4}|\omega_0|^2 \\ &+ \int_{B_{\delta_0}(p_n)} (U_{\delta,\varepsilon} - U_0)\frac{e^{2U_0}}{\rho^2}(|\nabla\chi_0|^2 + |\nabla\psi_0|^2). \end{aligned} \tag{4.70}$$

Note that this is justified since (3.13) implies that

$$\left|\int_{\partial B_{r_n}(p_n)} (U_{\delta,\varepsilon} - U_0)\partial_\nu U_0\right| \leq C\int_{\partial B_{r_n}(p_n)} \rho^{-1}|\log r_n| = Cr_n|\log r_n| \to 0 \quad \text{as} \quad r_n \to 0. \tag{4.71}$$

It follows that

$$\begin{aligned} \lim_{t\to 0}\frac{d}{dt}\mathcal{I}(\Psi^t_{\delta,\varepsilon}) =& \int_{\mathcal{C}_{\delta_0,\varepsilon_0}} 2\nabla U_0 \cdot \nabla(U_{\delta,\varepsilon} - U_0) + 4(U_{\delta,\varepsilon} - U_0)\frac{e^{4U_0}}{\rho^4}|\omega_0|^2 \\ &- \int_{\partial\mathcal{C}_{\delta_0,\varepsilon_0}} 2(U_{\delta,\varepsilon} - U_0)\partial_\nu U_0 + \int_{\mathcal{C}_{\delta_0,\varepsilon_0}} 2(U_{\delta,\varepsilon} - U_0)\frac{e^{2U_0}}{\rho^2}(|\nabla\chi_0|^2 + |\nabla\psi_0|^2). \end{aligned} \tag{4.72}$$

This in fact vanishes, since (4.70) holds with $B_{\delta_0}(p_n)$ replaced by $\mathcal{C}_{\delta_0,\varepsilon_0}$. Verification of this statement follows from

$$\left|\int_{\mathcal{C}_{\delta_0,\varepsilon_0}} \nabla U_0 \cdot \nabla(U_{\delta,\varepsilon} - U_0) + (U_{\delta,\varepsilon} - U_0)\Delta U_0\right| \leq \int_{\mathcal{C}_{\delta_0,\varepsilon_0}} |\nabla U_{\delta,\varepsilon}|^2 + |\nabla U_0|^2 + |\Delta U_0| \to 0 \quad \text{as} \quad \varepsilon_0 \to 0, \tag{4.73}$$

which is true since $U_{\delta,\varepsilon} = U$ and U_0 have finite reduced energy and $|\Delta U_0| \leq C\rho^{-2+\epsilon}$ for some $\epsilon > 0$ by (3.21). Hence (4.7) holds for $\Psi_{\delta,\varepsilon}$.

Now integrating (4.6) twice and applying the Gagliardo-Nirenberg-Sobolev inequality produces

$$\mathcal{I}(\Psi_{\delta,\varepsilon}) - \mathcal{I}(\Psi_0) \geq 2\int_{\mathbb{R}^3} |\nabla \operatorname{dist}_{\mathbb{H}^2_{\mathbb{C}}}(\Psi_{\delta,\varepsilon}, \Psi_0)|^2 dx \geq C\left(\int_{\mathbb{R}^3} \operatorname{dist}^6_{\mathbb{H}^2_{\mathbb{C}}}(\Psi_{\delta,\varepsilon}, \Psi_0)dx\right)^{\frac{1}{3}}. \tag{4.74}$$

By Proposition 4.5 $\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\mathcal{I}(\Psi_{\delta,\varepsilon}) = \mathcal{I}(\Psi)$, and thus in order to complete the proof it suffices to show that the limits may be passed under the integral on the right-hand side. By the triangle inequality, it is enough to show

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\int_{\mathbb{R}^3} \operatorname{dist}^6_{\mathbb{H}^2_{\mathbb{C}}}(\Psi_{\delta,\varepsilon}, \Psi)dx = 0. \tag{4.75}$$

As mentioned in Section 3, the geometry of complex hyperbolic space is invariant under the translations $\overline{v} = v + b\chi - c\psi$, $\overline{\chi} = \chi + c$, $\overline{\psi} = \psi + b$. Then using the triangle inequality and direct calculation produces

$$
\begin{aligned}
&\mathrm{dist}_{\mathbb{H}^2_{\mathbb{C}}}(\Psi_{\delta,\varepsilon},\Psi)\\
\leq& \mathrm{dist}_{\mathbb{H}^2_{\mathbb{C}}}((U_{\delta,\varepsilon},\overline{v}_{\delta,\varepsilon},\overline{\chi}_{\delta,\varepsilon},\overline{\psi}_{\delta,\varepsilon}),(U,\overline{v}_{\delta,\varepsilon},\overline{\chi}_{\delta,\varepsilon},\overline{\psi}_{\delta,\varepsilon}))+\mathrm{dist}_{\mathbb{H}^2_{\mathbb{C}}}((U,\overline{v}_{\delta,\varepsilon},\overline{\chi}_{\delta,\varepsilon},\overline{\psi}_{\delta,\varepsilon}),(U,\overline{v},\overline{\chi}_{\delta,\varepsilon},\overline{\psi}_{\delta,\varepsilon}))\\
&+\mathrm{dist}_{\mathbb{H}^2_{\mathbb{C}}}((U,\overline{v},\overline{\chi}_{\delta,\varepsilon},\overline{\psi}_{\delta,\varepsilon}),(U,\overline{v},\overline{\chi},\overline{\psi}_{\delta,\varepsilon}))+\mathrm{dist}_{\mathbb{H}^2_{\mathbb{C}}}((U,\overline{v},\overline{\chi},\overline{\psi}_{\delta,\varepsilon}),(U,\overline{v},\overline{\chi},\overline{\psi}))\\
\leq& C\left(|U-U_{\delta,\varepsilon}|+\frac{e^{2U}}{\rho^2}(|\overline{v}-\overline{v}_{\delta,\varepsilon}|+|\overline{\psi}_{\delta,\varepsilon}||\overline{\chi}-\overline{\chi}_{\delta,\varepsilon}|+|\overline{\chi}||\overline{\chi}-\overline{\chi}_{\delta,\varepsilon}|)+\frac{e^{U}}{\rho}(|\overline{\chi}-\overline{\chi}_{\delta,\varepsilon}|+|\overline{\chi}-\overline{\chi}_{\delta,\varepsilon}|)\right),
\end{aligned}
\tag{4.76}
$$

where $\overline{v}_{\delta,\varepsilon}=v_{\delta,\varepsilon}+b\chi_{\delta,\varepsilon}-c\psi_{\delta,\varepsilon}$ and similarly for $\overline{\chi}_{\delta,\varepsilon}$, $\overline{\psi}_{\delta,\varepsilon}$. Observe that

$$
\int_{\mathbb{R}^3}|U-U_{\delta,\varepsilon}|^6\leq\int_{\mathbb{R}^3\setminus B_{1/\delta}}|U-U_0|^6. \tag{4.77}
$$

Since U and U_0 are limits in $H^1(\mathbb{R}^3)$ of compactly supported functions, the Sobolev inequality implies that $U-U_0\in L^6(\mathbb{R}^3)$, and hence this integral converges to zero as $\delta\to 0$. Next, we have

$$
\int_{\mathbb{R}^3}\frac{e^{12U}}{\rho^{12}}|\overline{v}-\overline{v}_{\delta,\varepsilon}|^6\leq\int_{\mathbb{R}^3\setminus B_{1/\delta}}+\int_{\mathcal{C}_{\delta,\sqrt{\varepsilon}}}+\sum_{n=1}^{N}\int_{B_{2\delta}(p_n)}\frac{e^{12U}}{\rho^{12}}|\overline{v}-\overline{v}_0|^6. \tag{4.78}
$$

By Lemma 3.4 and (4.54)

$$
\begin{aligned}
\int_{\mathbb{R}^3\setminus B_{1/\delta}}\frac{e^{12U}}{\rho^{12}}|\overline{v}-\overline{v}_0|^6\leq& C\int_{\mathbb{R}^3\setminus B_{1/\delta}}\frac{1}{\rho^{12}}|\overline{v}-\overline{v}_0|^6\\
\leq& C\int_{\mathbb{R}^3\setminus B_{1/\delta}}\frac{|\overline{v}-\overline{v}_0|^4}{\rho^{10}}(|\nabla\overline{v}|^2+|\nabla\overline{v}_0|^2)\\
\leq& C\int_{\mathbb{R}^3\setminus B_{1/\delta}}\frac{e^{4U}}{\rho^4}|\nabla\overline{\omega}|^2+\frac{e^{4U_0}}{\rho^4}|\nabla\overline{\omega}_0|^2\\
&+C\int_{\mathbb{R}^3\setminus B_{1/\delta}}\frac{e^{2U}}{\rho^2}(|\nabla\overline{\chi}|^2+|\nabla\overline{\psi}|^2)+\frac{e^{2U_0}}{\rho^2}(|\nabla\overline{\chi}_0|^2+|\nabla\overline{\psi}_0|^2),
\end{aligned}
\tag{4.79}
$$

since $\rho^{-6}|\overline{v}-\overline{v}_0|^4$ is bounded. This integral then converges to zero as $\delta\to 0$, as each integrand appears in the reduced energy. Furthermore

$$
\int_{\mathcal{C}_{\delta,\sqrt{\varepsilon}}}\frac{e^{12U}}{\rho^{12}}|\overline{v}-\overline{v}_0|^6\leq C\int_{\mathcal{C}_{\delta,\sqrt{\varepsilon}}}\frac{|\overline{v}-\overline{v}_0|^6}{\rho^{12}}\leq C|\mathcal{C}_{\delta,\sqrt{\varepsilon}}|\to 0\quad\text{as}\quad\varepsilon\to 0, \tag{4.80}
$$

as (3.20) and (4.19) imply that $|\overline{v}-\overline{v}_0|\leq C\rho^2$ here.

Consider now an asymptotically cylindrical end represented by p_n. By choosing constants b and c (used to define $\overline{v}$) appropriately in certain domains, we may assume without loss of generality that $\overline{\chi}$, $\overline{\psi}$, $\overline{\chi}_0$, $\overline{\psi}_0$ vanish on the axis. Therefore we have that (3.14) implies $|\overline{v}-\overline{v}|_\Gamma|\leq Cr_n^{-2}\rho^2$ in $B_{2\delta}(p_n)$. Moreover (4.18) yields

$$
|\overline{v}-\overline{v}_0|\leq C\frac{\rho^2}{r_n^{5/2}}\quad\text{on}\quad B_{2\delta}(p_n), \tag{4.81}
$$

and similarly

$$
|\overline{\chi}|+|\overline{\psi}|+|\overline{\chi}_0|+|\overline{\psi}_0|\leq C\frac{\rho^2}{r_n^{3/2}}\quad\text{on}\quad B_{2\delta}(p_n). \tag{4.82}
$$

Next, using Lemma 3.4 produces

$$\begin{aligned}\int_{B_{2\delta}(p_n)}\frac{e^{12U}}{\rho^{12}}|\overline{v}-\overline{v}_0|^6 \leq & C\int_{B_{2\delta}(p_n)}\frac{r_n^{12}}{\rho^{12}}|\overline{v}-\overline{v}_0|^6\\ \leq & C\int_{B_{2\delta}(p_n)}\frac{r_n^{12}|\overline{v}-\overline{v}_0|^4}{\rho^{10}}\left(|\nabla\overline{v}|^2+|\nabla\overline{v}_0|^2\right)\\ \leq & C\int_{B_{2\delta}(p_n)}\frac{r_n^{8}|\overline{v}-\overline{v}_0|^4}{\rho^{6}}\left(\frac{e^{4U}}{\rho^4}|\overline{\omega}|^2+\frac{e^{4U_0}}{\rho^4}|\overline{\omega}_0|^2\right)\\ & +C\int_{B_{2\delta}(p_n)}\frac{r_n^{10}|\overline{v}-\overline{v}_0|^4}{\rho^{8}}\frac{e^{2U}}{\rho^2}\left(|\overline{\psi}|^2|\nabla\overline{\chi}|^2+|\overline{\chi}|^2|\nabla\overline{\psi}|^2\right)\\ & +C\int_{B_{2\delta}(p_n)}\frac{r_n^{10}|\overline{v}-\overline{v}_0|^4}{\rho^{8}}\frac{e^{2U_0}}{\rho^2}\left(|\overline{\psi}_0|^2|\nabla\overline{\chi}_0|^2+|\overline{\chi}_0|^2|\nabla\overline{\psi}_0|^2\right).\end{aligned} \tag{4.83}$$

From (4.81) and (4.82) it follows that

$$\frac{r_n^{8}|\overline{v}-\overline{v}_0|^4}{\rho^{6}}+\frac{r_n^{10}|\overline{v}-\overline{v}_0|^4}{\rho^{8}}\leq C, \tag{4.84}$$

and hence (4.83) may be estimated by reduced energies restricted to $B_{2\delta}(p_n)$, which converge to zero as $\delta\to 0$. Analogous arguments hold if p_n represents an asymptotically flat end. We conclude that (4.78) converges to zero.

Similar computations show that the remaining integrals arising from the right-hand side of (4.76) also converge to zero, and therefore (4.75) holds. □

Proof of Theorem 1.2. The asymptotic assumptions on the initial data (g,k,E,B) imply that (U,v,χ,ψ) satisfy the asymptotics (1.8)-(1.10), (4.8)-(4.11). Thus Theorem 4.1 applies, and Theorem 1.2 follows from (1.19) after setting

$$\mathcal{F}(\mathcal{J}_1,\ldots,\mathcal{J}_N,q_1^e,\ldots,q_N^e,q_1^b,\ldots,q_N^b)=\mathcal{M}(U_0,v_0,\chi_0,\psi_0). \tag{4.85}$$

□

Consider now Conjecture 1.3, and assume that equality is achieved in (1.24) for initial data (M,g,k,E,B) with $N>1$ black holes. If Ψ denotes the associated harmonic map data, then following the proof of Theorem 1.2 yields $\Psi\equiv\Psi_0$. Arguments in Section 2 suggest that (M,g,k,E,B) should then give rise to a stationary axisymmetric electrovacuum extremal black hole spacetime with disconnected horizon, and with g conformally flat. It is likely that this spacetime falls into the Israel-Wilson-Perjés class, which consists of solutions to the stationary (not necessarily axisymmetric) Einstein-Maxwell equations that are distinguished by having a conformally flat orbit space. Moreover, since the initial data set is maximal, it would then follow from [10] that such a spacetime must be the Majumdar-Papapetrou spacetime.

Appendix A. Revisiting the Heuristic Arguments

The heuristic physical arguments which motivate (1.1) go back to Penrose's original derivation of the Penrose inequality [21]. Typically in such arguments, it is assumed that the end state of gravitational collapse is a single Kerr-Newman black hole. However, a more appropriate assumption for the end state is a finite number of mutually distant Kerr-Newman black holes moving apart with asymptotically constant velocity. This should be the result, if for instance, two distant black holes were initially moving away from each other sufficiently fast. We will now describe the heuristic arguments for the mass-angular momentum-charge inequality in this setting. It appears that this has not been previously considered in the literature.

Let m_i, $\mathcal{J}_i$, q_i denote the ADM masses, angular momenta, and total charges of the end state black holes. Then the total (ADM) mass, angular momentum, and charge of the end state is $m = \sum m_i$, $\mathcal{J} = \sum \mathcal{J}_i$, $q = \sum q_i$. In a Kerr-Newman black hole these quantities satisfy the equation [14]

$$m_i^2 = \frac{A_i}{16\pi} + \frac{q_i^2}{2} + \frac{\pi(q_i^4 + 4\mathcal{J}_i^2)}{A_i}, \tag{A.1}$$

where A_i denotes horizon area. Moreover, as a function of A_i (keeping $\mathcal{J}_i$ and q_i fixed), the right-hand side is nondecreasing precisely when

$$A_i \geq 4\pi\sqrt{q_i^4 + 4\mathcal{J}_i^2}, \tag{A.2}$$

and this inequality is always satisfied with equality only for extreme black holes. Thus, computing the minimum value of the right-hand side of (A.1) yields

$$m_i^2 \geq \frac{q_i^2 + \sqrt{q_i^4 + 4\mathcal{J}_i^2}}{2}, \tag{A.3}$$

with equality only for extreme black holes. Let m_0, $\mathcal{J}_0$, q_0 denote the ADM mass, angular momentum, and total charge of an initial state. Under appropriate hypotheses, such as axisymmetry and the existence of a twist potential, angular momentum is conserved $\mathcal{J}_0 = \mathcal{J} = \sum \mathcal{J}_i$. Moreover, by assuming that no charged matter is present, the total charge is conserved $q_0 = q = \sum q_i$, and since gravitational waves may only carry away positive energy $m_0 \geq m = \sum m_i$.

Lemma A.1. *Let $a_i, b_i \in \mathbb{R}$ and let $a = \sum a_i$, $b = \sum b_i$. Then*

$$\left(a^4 + b^2\right)^{1/4} \leq \sum \left(a_i^4 + b_i^2\right)^{1/4}. \tag{A.4}$$

Proof. Let $c_i = |b_i|^{1/2}$ and $c = \sum c_i$, then

$$|b|^{1/2} \leq \left(\sum |b_i|\right)^{1/2} = \left(\sum c_i^2\right)^{1/2} \leq \sum c_i = c. \tag{A.5}$$

Hence $b^2 \leq c^4$. We conclude that

$$\left(a^4 + b^2\right)^{1/4} \leq \left(a^4 + c^4\right)^{1/4} \leq \sum \left(a_i^4 + c_i^4\right)^{1/4} = \sum \left(a_i^4 + b_i^2\right)^{1/4}. \tag{A.6}$$

□

Lemma A.2. *Let $a_i, b_i \in \mathbb{R}$ and let $a = \sum a_i$, $b = \sum b_i$. Then*

$$\sqrt{a^2 + \sqrt{a^4 + b^2}} \leq \sum \sqrt{a_i^2 + \sqrt{a_i^4 + b_i^2}}. \tag{A.7}$$

Proof. By Lemma A.1

$$\left(a^4 + b^2\right)^{1/2} \leq \left(\sum \left(a_i^4 + b_i^2\right)^{1/4}\right)^2. \tag{A.8}$$

Thus, it follows that

$$\sqrt{a^2 + \sqrt{a^4 + b^2}} \leq \sqrt{\left(\sum a_i\right)^2 + \left(\sum \left(a_i^4 + b_i^2\right)^{1/4}\right)^2} \leq \sum \sqrt{a_i^2 + \sqrt{a_i^4 + b_i^2}} \tag{A.9}$$

□

Now, let $a_i = q_i$, and $b_i = 2\mathcal{J}_i$, then we get

$$\sqrt{2}m = \sqrt{2}\sum m_i \geq \sum \sqrt{q_i^2 + \sqrt{q_i^4 + 4\mathcal{J}_i^2}} \geq \sqrt{q^2 + \sqrt{q^4 + 4\mathcal{J}^2}}. \tag{A.10}$$

Squaring both sides yields the desired result (1.1). We conclude that the heuristic arguments are sufficiently robust to support the mass-angular momentum-charge inequality, even for spacetimes with multiple black holes moving apart from one another at high velocities.

Appendix B. The Extreme Kerr-Newman and Majumdar-Papapetrou Harmonic Maps

First we record formulas for the extreme Kerr-Newman harmonic map. Recall that in Boyer-Lindquist coordinates the Kerr-Newmann metric takes the form

$$-\frac{\Delta - a^2\sin^2\theta}{\Sigma}dt^2 + \frac{2a\sin^2\theta}{\Sigma}\left(\widetilde{r}^2 + a^2 - \Delta\right)dtd\phi + \frac{(\widetilde{r}^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\sin^2\theta d\phi^2 + \frac{\Sigma}{\Delta}d\widetilde{r}^2 + \Sigma d\theta^2 \tag{B.1}$$

where

$$\Delta = \widetilde{r}^2 + a^2 + q^2 - 2m\widetilde{r}, \qquad \Sigma = \widetilde{r}^2 + a^2\cos^2\theta, \tag{B.2}$$

and the electromagnetic 4-potential is given by

$$\mathbf{A} = -\frac{q_e\widetilde{r}}{\Sigma}\left(dt + a\sin^2\theta d\phi\right) - \frac{q_b\cos\theta}{\Sigma}\left(adt + (\widetilde{r}^2 + a^2)d\phi\right), \tag{B.3}$$

The event horizon is located at the larger of the two solutions to the quadratic equation $\Delta = 0$, namely $\widetilde{r}_+ = m + \sqrt{m^2 - a^2 - q^2}$, where the angular momentum is given by $\mathcal{J} = ma$. For $\widetilde{r} > \widetilde{r}_+$ it holds that $\Delta > 0$, so that a new radial coordinate may be defined by

$$r = \frac{1}{2}(\widetilde{r} - m + \sqrt{\Delta}), \tag{B.4}$$

or rather

$$\begin{aligned} &\widetilde{r} = r + m + \frac{m^2 - a^2 - q^2}{4r}, \qquad m^2 \neq a^2 + q^2 \\ &\widetilde{r} = r + m, \qquad m^2 = a^2 + q^2. \end{aligned} \tag{B.5}$$

Note that the new coordinate is defined for $r > 0$, and a critical point for the right-hand side of (B.5) ($m^2 \neq a^2 + q^2$) occurs at the horizon, so that two isometric copies of the outer region are encoded on this interval. The coordinates (r, θ, ϕ) then form a (polar) Brill coordinate system, which is related to the (cylindrical) Brill coordinates via the usual transformation $\rho = r\sin\theta$, $z = r\cos\theta$. Finally, the harmonic map $(u_{\mathrm{KN}}, v_{\mathrm{KN}}, \chi_{\mathrm{KN}}, \psi_{\mathrm{KN}}) : \mathbb{R}^3 \setminus \Gamma \to \mathbb{H}^2_{\mathbb{C}}$, $U_{\mathrm{KN}} = u_{\mathrm{KN}} + \log\rho$, which determines the extreme Kerr-Newman solution is given by

$$\begin{aligned} u_{\mathrm{KN}} &= -\frac{1}{2}\log\left[\left(\widetilde{r}^2 + a^2 + \frac{a^2\sin^2\theta(2m\widetilde{r} - q^2)}{\Sigma}\right)\sin^2\theta\right], \\ v_{\mathrm{KN}} &= ma\cos\theta(3 - \cos^2\theta) - \frac{a(q^2\widetilde{r} - ma^2\sin^2\theta)\cos\theta\sin^2\theta}{\Sigma}, \\ \chi_{\mathrm{KN}} &= -\frac{qa\widetilde{r}\sin^2\theta}{\Sigma}, \\ \psi_{\mathrm{KN}} &= \frac{q(\widetilde{r}^2 + a^2)\cos\theta}{\Sigma}. \end{aligned} \tag{B.6}$$

The Euler-Lagrange equations satisfied by this and any other harmonic map $\Psi : \mathbb{R}^3 \to \mathbb{H}^2_{\mathbb{C}}$ are given by

$$\begin{aligned} \Delta u - 2e^{4u}|\nabla v + \chi\nabla\psi - \psi\nabla\chi|^2 - e^{2u}(|\nabla\chi|^2 + |\nabla\psi|^2) = 0, \\ \text{div}\left[e^{4u}(\nabla v + \chi\nabla\psi - \psi\nabla\chi)\right] = 0, \\ \text{div}(e^{2u}\nabla\chi) - 2e^{4u}\nabla\chi \cdot (\nabla v + \chi\nabla\psi - \psi\nabla\chi) = 0, \\ \text{div}(e^{2u}\nabla\psi) + 2e^{4u}\nabla\psi \cdot (\nabla v + \chi\nabla\psi - \psi\nabla\chi) = 0. \end{aligned} \tag{B.7}$$

Consider now the Majumdar-Papapetrou spacetime $\left(\mathbb{R} \times (\mathbb{R}^3 \setminus \cup_{n=1}^N p_n), ds^2\right)$ with

$$ds^2 = -f^{-2}dt^2 + f^2\delta, \qquad f = 1 + \sum_{n=1}^N \frac{m_n}{r_n}, \tag{B.8}$$

where $m_n = \sqrt{(q_n^e)^2 + (q_n^b)^2}$ represents the mass and total eletromagnetic charge of each black hole, δ is the Euclidean metric, and r_n is the Euclidean distance to each puncture. Axisymmetry may be imposed by choosing the punctures p_n to lie on the z-axis. Cylindrical coordinates (ρ, z, ϕ) in 3-space give rise to Brill coordinates with $U_{\text{MP}} = -\log f$, and the 4-potential is given by

$$\mathbf{A} = \kappa f dt + \sqrt{1-\kappa^2}\sum_{n=1}^N \frac{m_n(z - z_n)}{r_n} d\phi, \qquad 0 \le \kappa \le 1. \tag{B.9}$$

The constant κ relates the electric and magnetic charges to the mass by $q_n^e = \kappa m_n$ and $q_n^b = \sqrt{1-\kappa^2} m_n$. Typically the Majumdar-Papapetrou spacetime is stated without magnetic charges, however through a duality rotation

$$E = (\cos\vartheta)\tilde{E} - (\sin\vartheta)\tilde{B}, \qquad B = (\sin\vartheta)\tilde{E} + (\cos\vartheta)\tilde{B}, \tag{B.10}$$

magnetic charge may be introduced so that $\kappa = \cos\vartheta$. Since $E = \kappa\nabla\log f$ and $B = \sqrt{1-\kappa^2}\nabla\log f$, the electromagnetic potentials are obtained from (2.7)

$$d\chi_{\text{MP}} = \kappa\rho(\partial_z f d\rho - \partial_\rho f dz), \qquad d\psi_{\text{MP}} = \sqrt{1-\kappa^2}\rho(\partial_z f d\rho - \partial_\rho f dz), \tag{B.11}$$

so that

$$\chi_{\text{MP}} = \kappa\sum_{n=1}^N \frac{m_n(z - z_n)}{r_n}, \qquad \chi_{\text{MP}} = \sqrt{1-\kappa^2}\sum_{n=1}^N \frac{m_n(z - z_n)}{r_n}. \tag{B.12}$$

Lastly, since this spacetime is static there is no angular momentum, and hence $v_{\text{MP}} = 0$. This, combined with the fact that χ_{MP} and ψ_{MP} are proportional leads to a harmonic map with a 2-dimensional target that is isometric to hyperbolic space, namely $(u_{\text{MP}}, v_{\text{MP}}, \chi_{\text{MP}}, \psi_{\text{MP}}) : \mathbb{R}^3 \setminus \Gamma \to \mathbb{H}^2 \subset \mathbb{H}^2_{\mathbb{C}}$ where $U_{\text{MP}} = u_{\text{MP}} + \log\rho$.

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