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The Riemannian Penrose Inequality with Charge for Multiple Black Holes

Marcus Khuri, Gilbert Weinstein, and Sumio Yamada

Abstract. We present the outline of a proof of the Riemannian Penrose inequality with charge $r \le m + \sqrt{m^2 - q^2}$, where $A = 4\pi r^2$ is the area of the outermost apparent horizon with possibly multiple connected components, m is the total ADM mass, and q the total charge of a strongly asymptotically flat initial data set for the Einstein-Maxwell equations, satisfying the charged dominant energy condition, with no charged matter outside the horizon.

1. Introduction

In a seminal paper [**18**], R. Penrose examined the validity of the cosmic censorship conjecture, and outlined a heuristic argument which shows how using also Hawking's area theorem, [**10**], implies a related inequality. In [**17**], he generalized this heuristic argument leading to an inequality now referred to as the Penrose inequality. Consider a strongly asymptotically flat (SAF) Cauchy surface in a spacetime satisfying the dominant energy condition (DEC), with ADM mass m containing an event horizon of area $A = 4\pi r^2$, which undergoes gravitational collapse and settles to a Kerr solution. Since the ADM mass m_∞ of the final state is no greater than m, and since the area radius r_∞ is no less than r, and since for the final state we must have $m_\infty \ge \frac{1}{2} r_\infty$ in order to avoid naked singularities, we must have had $m \ge \frac{1}{2} r$ also at the beginning of the evolution. The event horizon is indiscernible in the original slice without knowing the full evolution. However, one may replace the event horizon by the outermost minimal area enclosure of the apparent horizon, the boundary of the region admitting trapped surfaces, and obtain the same inequality. A counterexample to the Penrose inequality would therefore have suggested data which leads under the Einstein evolution to naked singularities, while a proof of the inequality could be viewed as evidence in support of cosmic censorship.

The inequality further simplifies in the time-symmetric case, where the apparent horizon coincides with the outermost minimal area enclosure. The dominant energy condition reduces now to non-negative scalar curvature of the Cauchy hypersurface, leading to the Riemannian version of the inequality: the ADM mass and the area radius of the outermost compact minimal surface in a SAF 3-manifold

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of non-negative scalar curvature satisfy $m \geq \frac{r}{2}$ with equality if and only if the manifold is a Schwarzschild slice. Note that this characterizes the Schwarzschild slice as the unique minimizer of m among all such 3-manifolds admitting an outermost horizon of area $A = 4\pi r^2$.

This inequality was first proved by Huisken–Ilmanen [**11**] in the special case where the horizon is connected using the inverse mean curvature flow, an approach proposed by Jang–Wald [**13**], following Geroch [**6**] who had shown that the Hawking mass is non-decreasing under this flow. The inequality was proven in full generality by Bray [**1**] using a conformal flow of the initial Riemannian metric, and the positive mass theorem [**19**, **21**].

We now turn to the charged case which is slightly more subtle. It is natural to conjecture as above that the Reissner-Nordström spacetime (RN), the charged analog of Schwarzschild spacetime gives rise to the unique minimizer of m, given r and q. Since RN satisfies $m = \frac{1}{2}(r + q^2/r)$ where q is the total charge, one is thus led to conjecture that in any SAF data satisfying $R_g \geq 2(|E|^2 + |B|^2)$, where E and B are respectively the electric and magnetic field, and R_g is the scalar curvature of g, we have

$$m \geq \frac{1}{2}\left(r + \frac{q^2}{r}\right) \tag{1}$$

with equality if and only if the initial data is RN. This is shown in [**11**], based on Jang [**12**], but only for a connected horizon, since the proof is based on inverse mean curvature flow. In fact, (1) can fail if the horizon is not connected, and a counterexample based on Majumdar-Papapetrou (MP) initial data with two black holes was constructed in [**20**]. This counterexample, however, does not suggest a counterexample to cosmic censorship. This is because the right-hand side of (1) is not monotonically increasing in r. Indeed, already Jang observed that (1) is equivalent to two inequalities:

$$m - \sqrt{m^2 - q^2} \leq r \leq m + \sqrt{m^2 - q^2}. \tag{2}$$

Cosmic censorship suggests the upper bound always holds, while the counterexample in [**20**] violates the lower bound.

In this paper, we prove the upper bound in (2) for multiple black holes. By the positive mass theorem with charge we have $m \geq |q|$ with equality if and only if the data is MP [**8**]. Hence if $r \leq |q|$, the upper bound in (2) follows immediately

$$r \leq |q| \leq m \leq m + \sqrt{m^2 - q^2}.$$

It thus only remains to prove the upper bound under the additional hypothesis $|q| \leq r$. Under this hypothesis, it is the lower bound that follows immediately

$$m \leq |q| + \sqrt{m^2 - q^2} \leq r + \sqrt{m^2 - q^2}.$$

We note that the stability of the outermost horizon in fact implies $|q| \leq r$, provided the horizon is connected [**7**, **14**]. In view of the above, the upper bound in (2) is equivalent to (1) under the additional hypothesis $|q| \leq r$. The proof of this latter statement will be based on an adaptation of Bray's conformal flow; see also [**4**].

We now introduce a few definitions and state our main theorem and a corollary. A time-symmetric initial data set (M, g, E, B) consists of a 3-manifold M, a Riemannian metric g, and vector fields E and B. We assume that the data satisfies the Maxwell constraints with no charges outside the horizon $\operatorname{div}_g E = \operatorname{div}_g B = 0$, and the charged DEC

$$\mu = R_g - 2(|E|^2 + |B|^2) \geq 0. \tag{3}$$

We assume that the data is SAF, meaning that the complement of a compact set in M is the finite union of disjoint ends, and on each end the fields decay according to

$$g - \delta = O_2(|x|^{-1}), \quad E = O_1(|x|^{-2}), \quad B = O_1(|x|^{-2}),$$

and R_g is integrable. This guarantees that the ADM mass and the total electric and magnetic charges

$$m = \frac{1}{16\pi} \int_{S_\infty} (g_{ij,j} - g_{jj,i})\nu^i \, dA,$$
$$q_E = \frac{1}{4\pi} \int_{S_\infty} E_i \nu^i \, dA, \quad q_B = \frac{1}{4\pi} \int_{S_\infty} B_i \nu^i \, dA,$$

are well-defined. Here, ν is the outer unit normal, and the limit is taken in a designated end. Without loss of generality, we assume that the magnetic charge $q_B = 0$, and from now on denote $q = q_E$. This can always be achieved by a fixed rotation in the (E, B)-space. Conformally compactifying all but the designated end, we can now restrict our attention to surfaces which bound compact regions, and define S_2 to enclose S_1 to mean $S_1 = \partial\Omega_1$, $S_2 = \partial\Omega_2$ and $\Omega_1 \subset \Omega_2$. An *outermost horizon* is a compact minimal surface not enclosed in any other compact minimal surface.

THEOREM 1. *Let (M, g, E, B) be a SAF initial data set satisfying the chargeless Maxwell constraints, the charged DEC, with ADM mass m, total charge q, and admitting an outermost horizon of area $A = 4\pi r^2$. Then the upper bound in* (2) *holds with equality if and only if the data is RN. Suppose that $|q| \leq r$, then* (1) *holds with equality if and only if the data is RN.*

As noted above, the first statement follows from the second.

COROLLARY 1. *Given m and q, satisfying $m \geq |q|$, RN is the unique maximizer of A. Given A and q, satisfying $4\pi q^2 \leq A$, RN is the unique minimizer of m.*

We point out that the hypothesis of no charges outside the horizon seems necessary. On the one hand, our proof uses the divergence-free character of E and B in the final stage once we switch to inverse mean curvature flow. Indeed, we suspect that our conformal flow would not converge to Reissner-Nordström when charges are present outside the horizon. In fact, in [**15**], the authors conjecture that one could construct spherically symmetric counterexamples in this case. On the other hand, the heuristic argument based on cosmic censorship would not apply since matter can carry charges out to infinity leading to a final state with a total charge different from the initial state. Thus, without additional hypotheses, one is not able to say how the upper bound in (2) for the final state compares to the same expression for the initial state.

In what follows, a brief outline of the main elements in the proof of Theorem 1 is given. Full details will appear in a forthcoming paper.

2. The conformal flow

Consider a SAF initial data set (M_0, g_0, E_0, B_0) satisfying the Maxwell constraints and the charged DEC. We define the conformal flow

$$g_t = u_t^4 g_0, \quad E_t^i = u_t^{-6} E_0^i, \quad B_t^i = u_t^{-6} B_0^i, \quad u_0 = 1. \tag{4}$$

This immediately yields that the Maxwell constraints $\operatorname{div}_{g_t} E_t = \operatorname{div}_{g_t} B_t = 0$ are preserved under the flow and that the charge q_t is constant. The logarithmic velocity of the flow $v_t = \dot{u}_t / u_t$ is determined by the following elliptic problem

$$\Delta_{g_t} v_t - (|E_t|^2 + |B_t|^2) v_t = 0, \quad v_t \to -1 \text{ at } \infty, \quad v_t|_{\Sigma_t} = 0, \tag{5}$$

where Σ_t is the outermost horizon in g_t. We point out that by the maximum principle, $-1 < v < 0$; and by the Hopf boundary Lemma, the outward normal derivative of v on Σ_t is negative. In particular, this guarantees that the surfaces Σ_t always move outward. Using the covariance $L_g(vu) = u^5 L_{u^4 g} v$ of the conformal Laplacian $L_g = \Delta_g - \frac{1}{8} R_g$, we have

$$\frac{1}{8} \frac{d(u_t^5 R_{g_t})}{dt} = -L_{g_0} \dot{u}_t = -u_t^5 \left(|E_t|^2 + |B_t|^2 - \frac{1}{8} R_{g_t} \right) v_t;$$

hence, from (3), $u_t^4 \mu_t$ is constant, and in particular $\mu_t \geq 0$ for all t provided $\mu_0 \geq 0$. Thus the charged DEC is preserved. The proof of the existence of solutions to (4)–(5) follows [**1**] closely, and it is easily checked that A_t is constant. The remaining two ingredients of the proof are to show that the mass m_t is non-increasing, and the inequality (1) holds at some final time $T \in (0, \infty]$, implying that (1) holds also at the initial time $t = 0$.

3. Monotonicity

As in [**1**], the proof of monotonicity of m_t for our flow is based on a clever doubling argument by Bunting–Masood-Ul-Alam first introduced in [**2**]. However here a more judicious choice of conformal factor, inspired by [**16**], is required before we can apply the positive mass theorem. First, we note that since the flow (4)–(5) is autonomous, it is enough to show that $\dot{m}_t \leq 0$ at $t = 0$. For convenience we drop the subscript 0.

We take two copies $M_\pm$ of the exterior of Σ, attach them at Σ, and equip them with conformal metrics $g_\pm = w_\pm^4 g$, where $w_\pm = \frac{1}{2}\sqrt{(1 \pm v)^2 - \phi^2}$ and ϕ satisfies the differential inequality

$$\phi \left(\Delta_g \phi - \frac{\nabla v \cdot \nabla \phi}{v} \right) \geq \Lambda \left| |E|^2 + |B|^2 - \frac{|\nabla \phi|^2}{v^2} \right|, \tag{6}$$

for some $\Lambda > 0$ large enough, with boundary conditions $\partial_\nu \phi = 0$ on Σ, $\phi \to 0$ as $|x| \to \infty$. From the asymptotic expansion, it turns out that $|x|\phi \to |q|$ at infinity. Inequalities (3) and (6) guarantee that $R_{g_\pm} \geq 0$ if $\Lambda \geq 12$, and the boundary conditions guarantee that mean curvatures on both sides of the gluing agree. Furthermore the maximum principle, $m \geq |q|$, and the asymptotics of ϕ guarantee that $(1 \pm v)^2 - \phi^2 > 0$, and the asymptotics of $w_\pm$ guarantee that the M_+ end is compactified while the mass of the M_- end is given by $\widetilde{m} = m - \gamma$, where γ is determined by $v = -1 + \gamma/|x| + O(|x|^{-2})$. Since $v > -1$, we have $\gamma > 0$. The positive mass theorem [**19, 21**] can now be applied to conclude that $\widetilde{m} \geq 0$ with equality if and only if $(\widetilde{M}, g_{\pm})$ is the Euclidean space. Since, as in [**1**] $\dot{m} = 2(\gamma - m)$, we get monotonicity with equality if and only if $(\widetilde{M}, g_{\pm})$ is flat.

It remains to show that (6) has a positive solution satisfying the required boundary conditions. Since this part is very technical, we leave the details to our forthcoming article. The main idea is to solve (6) with equality replacing inequality on the exterior of a small neighborhood of the boundary $\Omega = \{x \in M \mid \operatorname{dist}(x, \Sigma) > \tau\}$. We use the Leray-Schauder fixed point theorem [**9**, Theorem 11.6] to accomplish this, with appropriately chosen Dirchlet boundary conditions on $\partial\Omega$. Using such a domain avoids the difficulty of singular coefficients that occurs at Σ due to the vanishing of v. Finally ϕ is then extended across $\partial\Omega$ while preserving the inequality (6). Although the regularity of the extended solution is only $C^{1,1}$ across $\partial\Omega$, this is enough for an application of the positive mass theorem as described in the preceding paragraph.

4. Exhaustion

Considerable effort is spent in [**1**] to show that the exterior of Σ_t converges as $t \to \infty$ to a Schwarzschild slice. We circumvent these difficulties and instead obtain (1) at a late time T. As in [**1**], we prove in two steps that the surface Σ_t eventually encloses any given compact surface. First, we show that no compact surface in M can enclose Σ_t for all t. Then we show that Σ_t must eventually enclose any given compact surface. It is here that the hypothesis $|q| \le r$ is used. Recall that this inequality is necesary for the connectedness of the outermost horizon. Thus at late times, Σ_T is connected, and hence the inverse mean curvature flow can be applied to obtain (1) for (M_T, g_T, E_T, B_T), where M_T is the exterior of Σ_T.

After a perturbation, it may be assumed that the initial data set (M, g, E, B) has charged harmonic asymptotics [**3**]. That is, in the asymptotic end, $g = U_0^4\delta$, $E = U_0^{-6}E_\delta$, $E_\delta = q\nabla r^{-1}$ where δ is the Euclidean metric, $R_g = -8U_0^{-5}\Delta_\delta U_0 = 2|E|^2$, and $B = 0$.

Lemma 1. *If $|q| < r$, then Σ_t cannot be entirely enclosed by the coordinate sphere $S_{r(t)}$ for all t, where $r(t) = \varepsilon r e^{2t}$ for some sufficiently small ε.*

Assume by contradiction that Σ_t is entirely enclosed by $S_{r(t)}$ for all t. We show that for some large T, Σ_T is not the outermost minimal area enclosure of Σ_0, yielding a contradiction.

Writing $U_t = u_t U_0$ and $V_t = v_t u_t U_0$, then we have

$$\Delta_\delta U_t = -\frac{1}{4}|E_\delta|^2 U_t^{-3}, \quad \Delta_\delta V_t = \frac{3}{4}U_t^{-4}|E_\delta|^2 V_t.$$

Let $\widetilde{V}_t$ be the unique solution of the second equation above with U_t replaced by $\widetilde{U}_t$, and satisfying $\widetilde{V}_t = 0$ on $S_{r(t)}$, and $\widetilde{V}_t \to -e^{-t}$ as $|x| \to \infty$, where $\widetilde{U}_t$ is the conformal factor U_t in the conformal flow of the Reissner-Nordström initial data. Note that $\widetilde{V}_t$ is the velocity $\tilde{v}_t\widetilde{U}_t$ in the conformal flow of the Reissner-Nordström initial data, where $\tilde{v}_t$ is obtained from (9) by setting $m^2 = 4e^{-4t}r(t)^2 + q^2$, and thus from (8)

$$\widetilde{U}_t = \left(e^{-2t} + \frac{\sqrt{4e^{-4t}r(t)^2 + q^2}}{|x|} + e^{-2t}\frac{r(t)^2}{|x|^2}\right)^{1/2}.$$

The idea is to compare V_t and $\widetilde{V}_t$ to obtain estimates on U_t in terms of $\widetilde{U}_t$. However, we only need to estimate $\int_{S_{r(t)}} U_t^4 d\sigma_\delta$. Thus, let $\widehat{U}_t$ be the unique solution of

$$\Delta_\delta \widehat{U}_t = -\frac{1}{4}|E_\delta|^2 \widehat{U}_t^{-3}, \quad \widehat{U}_t \to e^{-t} \text{ as } |x| \to \infty,$$
$$\widehat{U}_t|_{S_{r(t)}} = \left(\frac{1}{4\pi r(t)^2}\int_{S_{r(t)}} U_t^4\right)^{1/4}.$$

This radial function can be computed explicitly

(7)
$$\widehat{U}_t^4(x) = e^{-4t} + \frac{e^{-2t}\sqrt{\frac{8}{3}\left(\alpha+\frac{1}{2}q^2\right)}}{|x|} + \frac{\alpha}{|x|^2} + \frac{e^{2t}\sqrt{\frac{8}{3}\left(\alpha+\frac{1}{2}q^2\right)}(\alpha-q^2)}{6|x|^3} + \frac{e^{4t}(\alpha-q^2)^2}{36|x|^4},$$

where α is a positive constant depending on $\int_{S_{r(t)}} U_t^4$. The assumption $|q| \le r$ guarantees that $\alpha \ge q^2 + 6e^{-4t}r(t)^2$, and hence $\widehat{U}_t(x) \ge \widetilde{U}_t(x)$ for $|x| \ge r(t)$.

Now $W_t = \widetilde{V}_t - \widehat{V}_t$ satisfies

$$\Delta_\delta W_t = \frac{3}{4}\widehat{U}_t^{-4}|E_\delta|^2 W_t + \frac{3}{4}(\widetilde{U}_t^{-4} - \widehat{U}_t^{-4})\widetilde{V}_t|E_\delta|^2,$$

$W_t \to 0$ as $|x| \to \infty$, and $W_t > 0$ on $S_{r(t)}$ because $\widehat{V}_t(r(t)) = \frac{d}{dt}\widehat{U}_t(r(t)) < 0 = \widetilde{V}_t(r(t))$. Therefore, since $\widetilde{U}_t^{-4} - \widehat{U}_t^{-4} \ge 0$ the maximum principle gives that $W_t \ge 0$ outside $S_{r(t)}$.

This yields the upper bound $\widehat{V}_t \le \widetilde{V}_t$, and hence since $\widehat{V}_t = \frac{d}{dt}\widehat{U}_t$ it also gives an estimate of $\widehat{U}_t$ from above in terms of $\widetilde{V}_t$. This gives an upper bound on $\int_{S_{r(t)}} U_t^4$, and it then follows as in [**1**] that $|S_{r(t)}| \le \varepsilon^2 A[2 + O(\varepsilon^{-1}e^{-t})]^4$. Hence, for ε sufficiently small and T sufficiently large, we have $|S_{r(T)}| < A$, and Σ_T is not outer area minimizing, in contradiction to its definition.

5. Rigidity

In the case of equality, the mass $\widetilde{m}$ of the doubled manifold $(\widetilde{M}, g_\pm)$ in the monotonicity proof must be zero, hence $\widetilde{M}$ is $\mathbb{R}^3$ and consequently Σ is connected. Thus, we can use Disconzi-Khuri's definition of the charged Hawking mass

$$m_{\mathrm{CH}}(S) = \frac{r}{2}\left(1 - \frac{q^2}{r^2} - \frac{1}{16\pi}\int_S H^2\, dA\right),$$

and its monotonicity under the inverse mean curvature to show that if equality holds when the horizon is connected, then the initial data set is RN. Although B is assumed to vanish in [**5**, Theorem 1], the argument carries through in the time-symmetric case even if $B \ne 0$.

Finally, we note that if the initial data set is RN, then the conformal flow defined by (4) and (5) simply yields a rescaling of RN as indeed it must by rigidity. The RN metric can be written in isotropic coordinates as $-V^2dt^2 + g$, where $g = U^4\delta$,

$$U(x) = \left(1 + \frac{m}{|x|} + \frac{m^2 - q^2}{4|x|^2}\right)^{1/2}, \tag{8}$$

and the electric fields is $E_i = U^{-2}\partial_i(q/|x|)$. The conformal flow given by rescaling the coordinates $x \mapsto e^{-2t}x$ has logarithmic flow velocity

$$v_t = \frac{-e^{-2t} + e^{2t}(m^2 - q^2)/4|x|^2}{e^{-2t} + m/|x| + e^{2t}(m^2 - q^2)/4|x|^2}. \tag{9}$$

It is now straightforward to verify that v_t satisfies (4)–(5).

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Department of Mathematics, Stony Brook University, Stony Brook, New York 11794
E-mail address: khuri@math.sunysb.edu

Department of Physics and Department of Computer Sciences and Mathematics, Ariel University, Ariel 40700, Israel
E-mail address: gilbertw@ariel.ac.il

Department of Mathematics, Gakushuin University, Tokyo 171-8588, Japan
E-mail address: yamada@math.gakushuin.ac.jp